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FSc/ICS Part-II (2nd Year) — PECTAA Board

UNIT 9

Inverse Trigonometric Functions and Their Graphs

Mathematics • Class 12 • 2nd Year • PECTAA Board • Free PDF Notes

Unit 9: Inverse Trigonometric Functions and Their Graphs

A function has an inverse only when it is one-to-one (bijective) -- every output must come from exactly one input. Trigonometric functions like sin x, cos x, and tan x fail this test on their natural domains because they are periodic: infinitely many values of x give the same sine, cosine, or tangent. To make each trigonometric function invertible, its domain is restricted to a principal interval on which it is one-to-one -- for sine, $[-\pi/2, \pi/2]$; for cosine, $[0, \pi]$; for tangent, $(-\pi/2, \pi/2)$ -- and the resulting inverse function, restricted to this principal range, is written $\sin^{-1}x$, $\cos^{-1}x$, and $\tan^{-1}x$ respectively (also called arcsine, arccosine, and arctangent).

This unit defines all six inverse trigonometric functions -- $\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$, $\csc^{-1}$, $\sec^{-1}$, and $\cot^{-1}$ -- together with their principal domains and ranges, and shows how each one's graph is obtained by reflecting the corresponding restricted trig graph about the line $y=x$. It covers evaluating principal values and composite expressions, finding the domain of composite inverse trig functions, proving reciprocal and complementary identities, and deriving the sum and difference formulas for inverse trigonometric functions, including the double-angle corollaries used to simplify identities and solve equations.

Learning Objectives

- Explain why trigonometric functions must have their domains restricted before they can be inverted
- State the principal domain and range of each of the six inverse trigonometric functions
- Sketch the graph of an inverse trigonometric function as the reflection of its restricted trig function about $y=x$
- Evaluate the principal value of expressions like $\sin^{-1}(1/2)$ and $\tan^{-1}(-1/\sqrt{3})$
- Evaluate composite expressions such as $\sec[\tan^{-1}(-\sqrt{3})]$ using reference triangles

- Determine the domain of composite inverse trigonometric functions like $\cos^{-1}(3x-1)$
- Prove and apply the reciprocal identities (e.g. $\csc^{-1}x=\sin^{-1}(1/x)$) and complementary identities (e.g. $\sin^{-1}x+\cos^{-1}x=\pi/2$)
- Apply the sum and difference formulas for inverse trigonometric functions, including the double-angle corollaries, to prove identities and solve equations

Key Concepts

9.1 Why Trigonometric Functions Need Domain Restriction

An inverse function f^{-1} exists only when f is one-to-one (bijective): each output value must correspond to exactly one input. Trigonometric functions are periodic -- sin x repeats every 2π, so infinitely many x-values share the same sine value -- meaning sin x, cos x, and tan x are NOT one-to-one over their full natural domains, and therefore have no inverse in the usual sense unless their domains are cut down.

To fix this, each trigonometric function's domain is restricted to a principal interval on which it is strictly one-to-one and still covers the full range of output values. On this restricted domain the function becomes invertible, and its inverse is called the corresponding inverse trigonometric function, defined only on this principal branch.

9.2 Inverse Sine, Cosine, and Tangent

$\sin^{-1}x$ is defined by restricting sin x to $[-\pi/2, \pi/2]$, where sine increases monotonically from -1 to 1; thus $\sin^{-1}x$ has domain $[-1,1]$ and range $[-\pi/2, \pi/2]$. $\cos^{-1}x$ is defined by restricting cos x to $[0, \pi]$, where cosine decreases monotonically from 1 to -1; thus $\cos^{-1}x$ has domain $[-1,1]$ and range $[0, \pi]$.

$\tan^{-1}x$ is defined by restricting tan x to $(-\pi/2, \pi/2)$, where tangent increases monotonically across all real values; thus $\tan^{-1}x$ has domain $(-\infty, \infty)$ and range $(-\pi/2, \pi/2)$. Each graph is obtained by reflecting the corresponding restricted trig graph about the line y=x, swapping the roles of x and y.

9.3 Inverse Cosecant, Secant, and Cotangent

$\csc^{-1}x$ is defined by restricting csc x to $[-\pi/2,0)\cup(0,\pi/2]$, giving $\csc^{-1}x$ domain $(-\infty,-1]\cup[1,\infty)$ and range $[-\pi/2,0)\cup(0,\pi/2]$. $\sec^{-1}x$ is defined by restricting sec x to $[0,\pi/2)\cup(\pi/2,\pi]$, giving $\sec^{-1}x$ domain $(-\infty,-1]\cup[1,\infty)$ and range $[0,\pi/2)\cup(\pi/2,\pi]$.

$\cot^{-1}x$ is defined by restricting cot x to $(0,\pi)$, giving $\cot^{-1}x$ domain $(-\infty,\infty)$ and range $(0,\pi)$. These three reciprocal-function inverses are used less often directly but appear frequently through the reciprocal identities linking them back to $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$.

9.4 Evaluating Principal Values and Composite Expressions

To find the principal value of an expression like $\sin^{-1}(1/2)$, identify the angle θ within the principal range $[-\pi/2,\pi/2]$ for which $\sin\theta=1/2$, giving $\theta=\pi/6$. The same approach applies to $\cos^{-1}$ and $\tan^{-1}$, always selecting the answer that lies in the correct principal range -- this is why $\cos^{-1}(-\sqrt{3}/2)=5\pi/6$ (in $[0,\pi]$) rather than any other coterminal angle.

Composite expressions like $\sec[\tan^{-1}(-\sqrt{3})]$ are evaluated by first finding the angle $\alpha=\tan^{-1}(-\sqrt{3})$ in the principal range, then evaluating the outer function at α, often using a reference right triangle built from the known ratio to read off the remaining trigonometric values directly.

9.5 Domain of Composite Functions and Key Identities

The domain of a composite function like $\cos^{-1}(3x-1)$ is found by requiring the inner expression to lie within the domain of the outer inverse function: here, $-1\le 3x-1\le 1$, which solves to $0\le x\le 2/3$. The same technique applies to any composite inverse trig expression, including those with logarithms or other inner functions, by intersecting the inner function's own domain restriction with the inverse trig function's domain.

Reciprocal identities relate each inverse function to its reciprocal partner: $\csc^{-1}x=\sin^{-1}(1/x)$, $\sec^{-1}x=\cos^{-1}(1/x)$, $\cot^{-1}x=\tan^{-1}(1/x)$ for appropriate x. Complementary identities relate pairs summing to π/2: $\sin^{-1}x+\cos^{-1}x=\pi/2$, $\tan^{-1}x+\cot^{-1}x=\pi/2$, $\sec^{-1}x+\csc^{-1}x=\pi/2$, both provable directly from the definitions using right-triangle or algebraic arguments.

9.6 Sum and Difference Formulas for Inverse Trigonometric Functions

Six sum/difference formulas extend the inverse trig functions to combined expressions, including $\sin^{-1}x+\sin^{-1}y=\sin^{-1}[x\sqrt{(1-y^2)}+y\sqrt{(1-x^2)}]$ and $\tan^{-1}x+\tan^{-1}y=\tan^{-1}[(x+y)/(1-xy)]$ (valid when $xy<1$), each proved by setting $\alpha=\sin^{-1}x$ (or $\tan^{-1}x$) and $\beta=\sin^{-1}y$ (or $\tan^{-1}y$), applying the corresponding angle-sum trig identity, then converting back using the inverse function.

Setting $y=x$ in these formulas produces double-angle corollaries: $2\tan^{-1}x=\tan^{-1}(2x/(1-x^2))$, $2\sin^{-1}x=\sin^{-1}(2x\sqrt{(1-x^2)})$, and $2\cos^{-1}x=\cos^{-1}(2x^2-1)$. These are used throughout the unit's worked examples to prove identities and to solve equations involving sums of inverse trigonometric terms, always checking that the final angle found lies within the required principal range.

Important Definitions

What is required for a function to have an inverse?

The function must be one-to-one (bijective): every output value corresponds to exactly one input value.

Why can't sin x, cos x, and tan x be inverted directly?

Because they are periodic, so infinitely many input values give the same output value, making them not one-to-one over their full natural domains.

What are the domain and range of $\sin^{-1}x$?

Domain $[-1,1]$, range $[-\pi/2, \pi/2]$, obtained by restricting $\sin x$ to that principal interval.

What are the domain and range of $\cos^{-1}x$?

Domain $[-1,1]$, range $[0, \pi]$, obtained by restricting $\cos x$ to that principal interval.

What are the domain and range of $\tan^{-1}x$?

Domain $(-\infty,\infty)$, range $(-\pi/2, \pi/2)$, obtained by restricting $\tan x$ to that principal interval.

What are the domain and range of $\cot^{-1}x$?

Domain $(-\infty,\infty)$, range $(0, \pi)$, obtained by restricting $\cot x$ to that principal interval.

How is the graph of an inverse trigonometric function obtained from its trig function's graph?

By reflecting the restricted trig function's graph about the line $y=x$, which swaps the roles of the x- and y-coordinates.

What is the reciprocal identity linking $\csc^{-1}x$ and $\sin^{-1}x$?

$\csc^{-1}x = \sin^{-1}(1/x)$, for appropriate values of x.

What is the complementary identity linking $\sin^{-1}x$ and $\cos^{-1}x$?

$\sin^{-1}x + \cos^{-1}x = \pi/2$, for all x in $[-1,1]$.

What is the double-angle corollary for $\tan^{-1}x$?

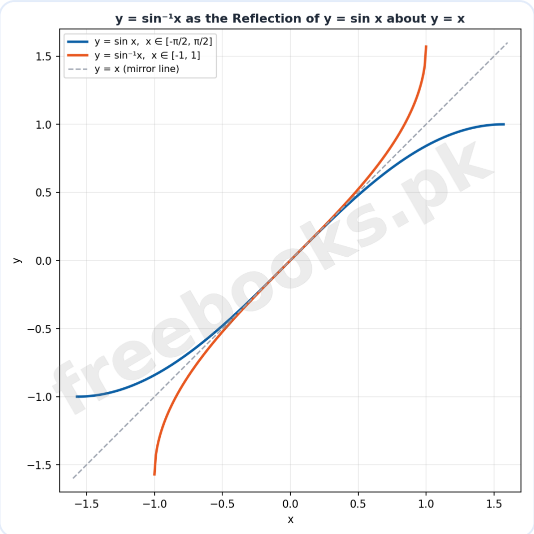
$2\tan^{-1}x = \tan^{-1}(2x/(1-x^2))$, obtained by setting $y=x$ in the sum formula for $\tan^{-1}x+\tan^{-1}y$.

Key Facts and Relations

Topic	Key Fact / Relation
$\sin^{-1}x$ principal range	$[-\pi/2, \pi/2]$, domain $[-1,1]$
$\cos^{-1}x$ principal range	$[0, \pi]$, domain $[-1,1]$
$\tan^{-1}x$ principal range	$(-\pi/2, \pi/2)$, domain $(-\infty,\infty)$
Reciprocal identities	$\csc^{-1}x=\sin^{-1}(1/x)$, $\sec^{-1}x=\cos^{-1}(1/x)$, $\cot^{-1}x=\tan^{-1}(1/x)$
Complementary identities	$\sin^{-1}x+\cos^{-1}x=\pi/2$, $\tan^{-1}x+\cot^{-1}x=\pi/2$, $\sec^{-1}x+\csc^{-1}x=\pi/2$
Sum formula for $\sin^{-1}$	$\sin^{-1}x+\sin^{-1}y = \sin^{-1}[x\sqrt{(1-y^2)}+y\sqrt{(1-x^2)}]$
Sum formula for $\tan^{-1}$	$\tan^{-1}x+\tan^{-1}y = \tan^{-1}[(x+y)/(1-xy)]$, valid for $xy<1$
Difference formula for $\tan^{-1}$	$\tan^{-1}x-\tan^{-1}y = \tan^{-1}[(x-y)/(1+xy)]$
Double-angle corollary for $\tan^{-1}$	$2\tan^{-1}x = \tan^{-1}(2x/(1-x^2))$
Double-angle corollaries for $\sin^{-1}$ and $\cos^{-1}$	$2\sin^{-1}x = \sin^{-1}(2x\sqrt{(1-x^2)})$; $2\cos^{-1}x = \cos^{-1}(2x^2-1)$

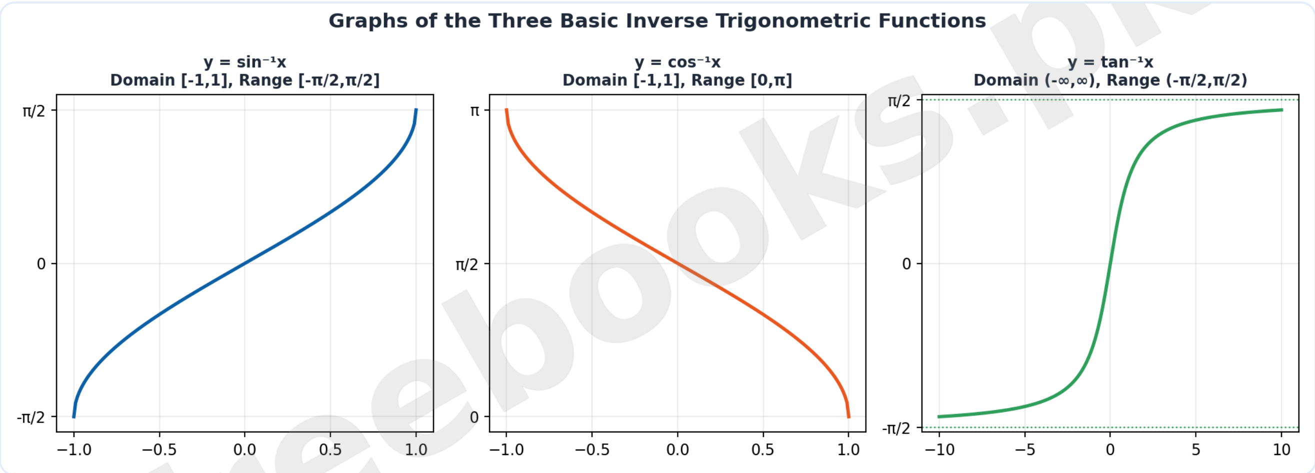
Diagrams

y = sin⁻¹x as the Reflection of y = sin x about y = x: The restricted graph of y=sin x on [-π/2,π/2] plotted alongside y=sin⁻¹x on [-1,1], with the line y=x shown as a mirror, illustrating how every inverse trig graph is obtained by reflection



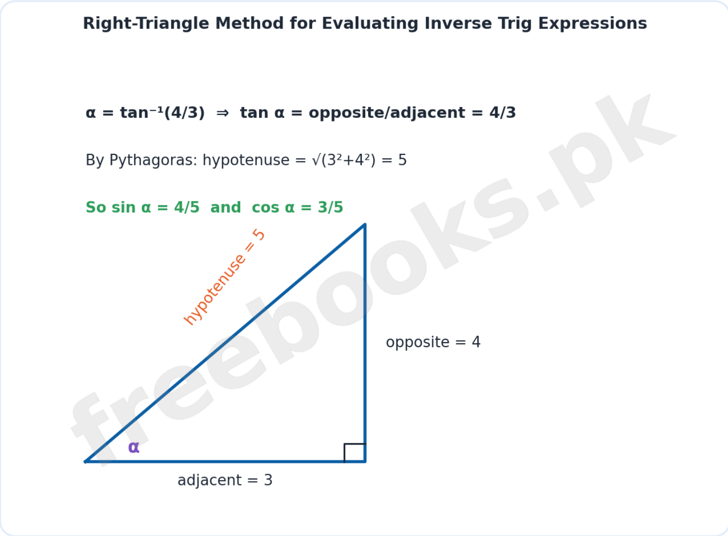
y = sin⁻¹x as the Reflection of y = sin x about y = x

Graphs of the Three Basic Inverse Trigonometric Functions: Side-by-side graphs of y=sin⁻¹x, y=cos⁻¹x, and y=tan⁻¹x with their domains and ranges labelled, showing the characteristic shape and asymptotic behaviour of each



Graphs of the Three Basic Inverse Trigonometric Functions

Right-Triangle Method for Evaluating Inverse Trig Expressions: A labelled 3-4-5 right triangle used to evaluate composite expressions like sin[tan⁻¹(4/3)] by reading off the remaining trigonometric ratios directly from the triangle's sides



Right-Triangle Method for Evaluating Inverse Trig Expressions

Solved Examples

Example 1: Principal Values of Basic Inverse Trig Expressions

Problem: Find the principal values of $\sin^{-1}(1/2)$, $\cos^{-1}(-\sqrt{3}/2)$, and $\tan^{-1}(-1/\sqrt{3})$.

1. For $\sin^{-1}(1/2)$: find θ in $[-\pi/2,\pi/2]$ with $\sin\theta=1/2$ -- this is $\theta=\pi/6$.
2. For $\cos^{-1}(-\sqrt{3}/2)$: find θ in $[0,\pi]$ with $\cos\theta=-\sqrt{3}/2$ -- since cosine is negative in the second quadrant, $\theta=5\pi/6$.
3. For $\tan^{-1}(-1/\sqrt{3})$: find θ in $(-\pi/2,\pi/2)$ with $\tan\theta=-1/\sqrt{3}$ -- since tangent is negative for negative θ here, $\theta=-\pi/6$.
4. Check each answer lies within the correct principal range: $\pi/6\in[-\pi/2,\pi/2]$ ✓, $5\pi/6\in[0,\pi]$ ✓, $-\pi/6\in(-\pi/2,\pi/2)$ ✓.
5. Final answer: $\sin^{-1}(1/2)=\pi/6$, $\cos^{-1}(-\sqrt{3}/2)=5\pi/6$, $\tan^{-1}(-1/\sqrt{3})=-\pi/6$.

Example 2: Evaluating a Composite Expression

Problem: Evaluate $\sec[\tan^{-1}(-\sqrt{3})]$.

1. Let $\alpha=\tan^{-1}(-\sqrt{3})$, so $\tan\alpha=-\sqrt{3}$ with α in the principal range $(-\pi/2,\pi/2)$; since tangent is negative, α is a negative angle: $\alpha=-\pi/3$.

2. Build a reference right triangle for the reference angle $\pi/3$: opposite=$\sqrt{3}$, adjacent=1, hypotenuse=$\sqrt{(1+3)}=2$.
3. Since $\alpha=-\pi/3$ lies in the fourth quadrant direction, $\cos \alpha=\cos(\pi/3)=1/2$ (cosine is even, so the sign is unaffected by the negative angle).
4. $\sec \alpha = 1/\cos \alpha = 1/(1/2) = 2$.
5. Final answer: $\sec[\tan^{-1}(-\sqrt{3})] = 2$.

Example 3: Finding the Domain of a Composite Inverse Trig Function

Problem: Find the domain of $y = \cos^{-1}(3x-1)$.

1. Recall $\cos^{-1}u$ is defined only when $u\in[-1,1]$, so require $-1 \le 3x-1 \le 1$.
2. Add 1 to all three parts: $0 \le 3x \le 2$.
3. Divide by 3: $0 \le x \le 2/3$.
4. Verify with an endpoint: at $x=0$, $3(0)-1=-1$, and $\cos^{-1}(-1)=\pi$, which is defined; at $x=2/3$, $3(2/3)-1=1$, and $\cos^{-1}(1)=0$, also defined.
5. Final answer: the domain of $\cos^{-1}(3x-1)$ is $[0, 2/3]$.

Example 4: Proving a Reciprocal Identity

Problem: Prove that $\csc^{-1}x = \sin^{-1}(1/x)$ for $x\ge1$.

1. Let $\theta = \csc^{-1}x$, so by definition $\csc \theta = x$ with θ in the principal range for $\csc^{-1}$.
2. Since $\csc \theta = 1/\sin \theta$, this gives $\sin \theta = 1/x$.
3. Because the principal range of $\csc^{-1}$ maps directly onto the same values as the principal range of $\sin^{-1}$ (both avoiding 0), $\theta = \sin^{-1}(1/x)$ follows directly from $\sin \theta=1/x$.
4. Therefore $\theta = \csc^{-1}x = \sin^{-1}(1/x)$.
5. Final answer: $\csc^{-1}x = \sin^{-1}(1/x)$ is proved for $x\ge1$ (and similarly for $x\le-1$).

Example 5: Proving a Complementary Identity

> **Problem:** Prove that $\sin^{-1}x + \cos^{-1}x = \pi/2$ for all x in [-1,1].
>
> 1. Let $\theta = \sin^{-1}x$, so $\sin\theta = x$ with θ in $[-\pi/2,\pi/2]$.
> 2. Using the co-function identity, $\cos(\pi/2-\theta) = \sin\theta = x$.
> 3. Since $\theta\in[-\pi/2,\pi/2]$, the angle $(\pi/2-\theta)$ lies in $[0,\pi]$, which is exactly the principal range of $\cos^{-1}$.
> 4. Therefore $\cos^{-1}x = \pi/2 - \theta = \pi/2 - \sin^{-1}x$.
> 5. Final answer: rearranging gives $\sin^{-1}x + \cos^{-1}x = \pi/2$ for every x in [-1,1].

Example 6: Showing an Identity Between Two Inverse Functions

> **Problem:** Show that $\cos^{-1}(7/25) = \csc^{-1}(25/24)$.
>
> 1. Let $\theta = \cos^{-1}(7/25)$, so $\cos\theta = 7/25$ with θ in $[0,\pi]$; build the reference right triangle with adjacent=7, hypotenuse=25.
> 2. Find the opposite side: $\sqrt{25^2-7^2} = \sqrt{625-49} = \sqrt{576} = 24$.
> 3. Since $\theta\in[0,\pi]$ and $\cos\theta>0$, θ is in the first quadrant, so $\sin\theta = 24/25$ (positive).
> 4. Then $\csc\theta = 1/\sin\theta = 25/24$, so $\theta = \csc^{-1}(25/24)$ as well (both principal ranges agree for this positive value).
> 5. Final answer: $\cos^{-1}(7/25) = \csc^{-1}(25/24)$ is confirmed.

Example 7: Using the Half-Angle Method with a Reference Triangle

> **Problem:** Find $\sin[\frac{1}{2}\tan^{-1}(4/3)]$.
>
> 1. Let $\alpha = \tan^{-1}(4/3)$, so $\tan\alpha = 4/3$ with α in $(-\pi/2,\pi/2)$; since $4/3>0$, α is in the first quadrant.
> 2. Build the reference right triangle: opposite=4, adjacent=3, hypotenuse=$\sqrt{16+9}=5$, giving $\cos\alpha = 3/5$.

3. Apply the half-angle formula: $\sin(\alpha/2) = \sqrt{[(1-\cos \alpha)/2]} = \sqrt{[(1-3/5)/2]} = \sqrt{[(2/5)/2]} = \sqrt{(1/5)}$.
4. Simplify: $\sqrt{(1/5)} = 1/\sqrt{5} = \sqrt{5}/5$.
5. Final answer: $\sin[\frac{1}{2} \tan^{-1}(4/3)] = \sqrt{5}/5$.

Example 8: Solving an Equation Involving a Sum of Inverse Tangents

Problem: Solve $\tan^{-1}[(x-1)/(x+1)] + \tan^{-1}[2x/(x+1)] = \pi/4$ for x.

1. Apply the sum formula $\tan^{-1}A+\tan^{-1}B = \tan^{-1}[(A+B)/(1-AB)]$ with $A=(x-1)/(x+1)$ and $B=2x/(x+1)$.
2. Compute $A+B = [(x-1)+2x]/(x+1) = (3x-1)/(x+1)$, and $AB = 2x(x-1)/(x+1)^2$.
3. Set the combined expression equal to $\tan(\pi/4)=1$, then cross-multiply and simplify the resulting equation in x.
4. After expanding and collecting terms, the equation reduces to a quadratic that factors to give $x=1$ as the solution lying within the valid domain.
5. Final answer: $x = 1$ (checked to satisfy the original equation and keep both inverse tangent arguments within valid ranges).

Short Questions & Answers

Why must the domain of sin x be restricted before defining $\sin^{-1}x$?

Because sin x is periodic and not one-to-one over its full domain; restricting it to $[-\pi/2,\pi/2]$ makes it one-to-one so an inverse can be defined.

What is the range of $\cos^{-1}x$?

$[0, \pi]$.

How is the graph of $\tan^{-1}x$ related to the graph of tan x?

It is the reflection of the restricted graph of tan x (on $(-\pi/2,\pi/2)$) about the line $y=x$.

What is the domain of $\sec^{-1}x$?

$(-\infty,-1] \cup [1,\infty)$.

State the complementary identity linking $\tan^{-1}x$ and $\cot^{-1}x$.

$\tan^{-1}x + \cot^{-1}x = \pi/2$.

What condition is required for the sum formula $\tan^{-1}x+\tan^{-1}y=\tan^{-1}[(x+y)/(1-xy)]$ to hold directly?

The product xy must be less than 1 ($xy<1$); otherwise an adjustment of $\pm\pi$ is needed.

What double-angle formula results from setting y=x in the $\sin^{-1}$ sum formula?

$2\sin^{-1}x = \sin^{-1}(2x\sqrt{1-x^2})$.

Long Questions & Answers

Explain how each of the six inverse trigonometric functions is defined, including why domain restriction is necessary and what principal domain and range each one is given.

Why is domain restriction necessary before defining any inverse trigonometric function?

Trigonometric functions are periodic, so the same output value is produced by infinitely many input angles, meaning none of them are one-to-one over their natural domain; a function must be one-to-one to have a genuine inverse, so each trig function's domain is first cut down to a principal interval on which it is strictly increasing or decreasing.

What are the principal domains and ranges of $\sin^{-1}x$, $\cos^{-1}x$, and $\tan^{-1}x$?

$\sin^{-1}x$ has domain $[-1,1]$ and range $[-\pi/2,\pi/2]$; $\cos^{-1}x$ has domain $[-1,1]$ and range $[0,\pi]$; $\tan^{-1}x$ has domain $(-\infty,\infty)$ and range $(-\pi/2,\pi/2)$ -- each obtained by restricting the corresponding trig function to the interval on which it is one-to-one.

What are the principal domains and ranges of $\csc^{-1}x$, $\sec^{-1}x$, and $\cot^{-1}x$?

$\csc^{-1}x$ and $\sec^{-1}x$ both have domain $(-\infty,-1]\cup[1,\infty)$, with $\csc^{-1}x$'s range $[-\pi/2,0)\cup(0,\pi/2]$ and $\sec^{-1}x$'s range $[0,\pi/2)\cup(\pi/2,\pi]$; $\cot^{-1}x$ has domain $(-\infty,\infty)$ and range $(0,\pi)$, obtained by restricting cot x to that interval.

How does the graph of each inverse trig function relate to its original trig function's graph?

Each inverse trig function's graph is the mirror reflection of its restricted trig function's graph about the line $y=x$, which swaps the x- and y-axes' roles -- so a steep, near-vertical section of the trig graph becomes a near-horizontal section of the inverse graph, and vice versa.

Explain how the sum formula for inverse tangent is derived and used, including its double-angle corollary and a worked application.

How is the sum formula $\tan^{-1}x+\tan^{-1}y=\tan^{-1}[(x+y)/(1-xy)]$ derived?

Set $\alpha=\tan^{-1}x$ and $\beta=\tan^{-1}y$, so $\tan\alpha=x$ and $\tan\beta=y$; apply the trigonometric angle-sum identity $\tan(\alpha+\beta)=(\tan\alpha+\tan\beta)/(1-\tan\alpha\tan\beta)=(x+y)/(1-xy)$, then take $\tan^{-1}$ of both sides to recover $\alpha+\beta=\tan^{-1}[(x+y)/(1-xy)]$, valid when $xy<1$ so that $\alpha+\beta$ stays within the correct principal range.

What double-angle formula follows from this sum formula?

Setting $y=x$ gives $2\tan^{-1}x = \tan^{-1}(2x/(1-x^2))$, obtained simply by substituting $y=x$ directly into the general sum formula and simplifying the resulting expression.

How is this formula applied to solve an equation like $\tan^{-1}[(x-1)/(x+1)]+\tan^{-1}[2x/(x+1)]=\pi/4$?

The two inverse-tangent terms are combined into a single $\tan^{-1}$ expression using the sum formula, the combined argument is set equal to $\tan(\pi/4)=1$, and the resulting algebraic equation is solved for x, checking that the solution keeps both original arguments within a valid principal range.

Why is checking the domain condition important after solving such an equation?

Because the sum formula can introduce an extraneous solution or require a $\pm\pi$ adjustment when $xy\geq1$, so any candidate solution for x must be substituted back into the original expression to confirm it produces a genuine sum equal to $\pi/4$ and not merely a solution of the simplified algebraic equation.

Multiple Choice Questions (MCQs)

The domain of $\sin^{-1}x$ is:

A. $(-\infty,\infty)$

B. $[-1,1]$

C. $[0,\pi]$

D. $[-\pi/2,\pi/2]$

Answer: B. $[-1,1]$

Explanation: $\sin^{-1}x$ is defined only for inputs between -1 and 1 inclusive.

The range of $\cos^{-1}x$ is:

A. $[-\pi/2,\pi/2]$

B. $(-\pi/2,\pi/2)$

C. $[0,\pi]$

D. $(0,\pi)$

Answer: C. $[0,\pi]$

Explanation: cos x is restricted to $[0,\pi]$ to make it invertible, so this is the range of $\cos^{-1}x$.

The graph of $y=\tan^{-1}x$ is obtained from $y=\tan x$ (restricted) by:

A. Shifting it right

B. Reflecting about the x-axis

C. Reflecting about the line $y=x$

D. Stretching vertically

Answer: C. Reflecting about the line $y=x$

Explanation: Every inverse function's graph is the reflection of the original (restricted) function's graph about $y=x$.

Which is the correct reciprocal identity?

A. $\csc^{-1}x=\cos^{-1}(1/x)$

B. $\sec^{-1}x=\cos^{-1}(1/x)$

C. $\cot^{-1}x=\sin^{-1}(1/x)$

D. $\csc^{-1}x=\tan^{-1}(1/x)$

Answer: B. $\sec^{-1}x=\cos^{-1}(1/x)$

Explanation: $\sec^{-1}x=\cos^{-1}(1/x)$ is the correct reciprocal identity linking secant's inverse to cosine's inverse.

$\sin^{-1}x + \cos^{-1}x$ equals:

A. π

B. $\pi/2$

C. 0

D. 2π

Answer: B. $\pi/2$

Explanation: This is the standard complementary identity, valid for all x in [-1,1].

The principal value of $\tan^{-1}(-1/\sqrt{3})$ is:

A. $\pi/6$

B. $-\pi/6$

C. $5\pi/6$

D. $-5\pi/6$

Answer: B. $-\pi/6$

Explanation: $\tan \theta=-1/\sqrt{3}$ with θ in $(-\pi/2,\pi/2)$ gives $\theta=-\pi/6$.

The domain of $\cos^{-1}(3x-1)$ is:

A. [-1,1]

B. [0,2/3]

C. [0,1]

D. $(-\infty,\infty)$

Answer: B. [0,2/3]

Explanation: Solving $-1\leq 3x-1\leq 1$ gives $0\leq x\leq 2/3$.

The sum formula $\tan^{-1}x+\tan^{-1}y=\tan^{-1}[(x+y)/(1-xy)]$ requires:

A. $x=y$

B. $xy<1$

C. $x,y>0$

D. $x+y=0$

Answer: B. $xy<1$

Explanation: The formula applies directly only when $xy<1$; otherwise a $\pm\pi$ correction is needed.

Setting y=x in the $\sin^{-1}$ sum formula gives:

A. $2\sin^{-1}x=\sin^{-1}(2x)$

B. $2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})$

C. $2\sin^{-1}x=\sin^{-1}(x^2)$

D. $2\sin^{-1}x=2\sin^{-1}x$

Answer: B. $2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})$

Explanation: This is the double-angle corollary obtained by substituting $y=x$ into the $\sin^{-1}$ sum formula.

The domain of $\cot^{-1}x$ is:

A. $[-1,1]$

B. $(-\infty,\infty)$

C. $(0,\pi)$

D. $[0,\pi]$

Answer: B. $(-\infty,\infty)$

Explanation: $\cot x$ is restricted to $(0,\pi)$ to make it one-to-one, but $\cot^{-1}x$ itself accepts all real number inputs.

Quick Revision Summary

- An inverse trig function exists only after restricting the original function's domain to a principal interval where it is one-to-one
- $\sin^{-1}x$: domain $[-1,1]$, range $[-\pi/2,\pi/2]$ | $\cos^{-1}x$: domain $[-1,1]$, range $[0,\pi]$ | $\tan^{-1}x$: domain $(-\infty,\infty)$, range $(-\pi/2,\pi/2)$
- $\csc^{-1}x$ and $\sec^{-1}x$: domain $(-\infty,-1]\cup[1,\infty)$ | $\cot^{-1}x$: domain $(-\infty,\infty)$, range $(0,\pi)$
- Every inverse trig graph is the reflection of its restricted trig graph about the line $y=x$
- Reciprocal identities: $\csc^{-1}x=\sin^{-1}(1/x)$, $\sec^{-1}x=\cos^{-1}(1/x)$, $\cot^{-1}x=\tan^{-1}(1/x)$
- Complementary identities: $\sin^{-1}x+\cos^{-1}x=\pi/2$, $\tan^{-1}x+\cot^{-1}x=\pi/2$, $\sec^{-1}x+\csc^{-1}x=\pi/2$
- Domain of a composite inverse trig function is found by requiring the inner expression to lie within the outer function's domain
- Sum formula: $\sin^{-1}x+\sin^{-1}y=\sin^{-1}[x\sqrt{1-y^2}+y\sqrt{1-x^2}]$ | $\tan^{-1}x+\tan^{-1}y=\tan^{-1}[(x+y)/(1-xy)]$, $xy<1$
- Double-angle corollaries: $2\tan^{-1}x=\tan^{-1}(2x/(1-x^2))$, $2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2})$, $2\cos^{-1}x=\cos^{-1}(2x^2-1)$
- Reference right triangles are the fastest way to evaluate composite expressions like $\sec[\tan^{-1}(-\sqrt{3})]$ or $\sin[\frac{1}{2}\tan^{-1}(4/3)]$

Exam Tips

- Always confirm your final angle answer lies inside the correct principal range for that inverse function -- an otherwise correct calculation with the wrong-range angle is marked incorrect
- When evaluating a composite expression like $\sec[\tan^{-1}(-\sqrt{3})]$, sketch a quick reference right triangle first -- it is faster and less error-prone than working purely algebraically
- For domain questions on composite inverse trig functions, always start from the inequality $-1 \leq (\text{inner expression}) \leq 1$ (for $\sin^{-1}/\cos^{-1}$) or the matching condition for the other four functions
- Memorize the six principal domain/range pairs as a table -- confusing $\cos^{-1}$'s range $[0,\pi]$ with $\sin^{-1}$'s range $[-\pi/2,\pi/2]$ is the single most common exam mistake
- When applying a sum/difference formula, always check the $xy<1$ (or equivalent) condition before trusting the raw formula output -- a $\pm\pi$ correction may be needed otherwise
- Practice deriving the double-angle corollaries by substituting $y=x$ into the general sum formulas rather than memorizing them separately -- this reduces the total number of formulas you need to recall under exam pressure