

Class 9 · PCTB Board · English Medium

Mathematics Class 9

Complete Notes — All 13 Units

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Unit 1: Real Numbers

Numbers developed over thousands of years, from the Sumerian base-60 system through Egyptian, Roman, and Indian numerals to the modern Arabic decimal system used worldwide today. The set of real numbers R is defined as the union of two disjoint sets: the rational numbers Q (which can be written as p/q with integers p , q and $q \neq 0$) and the irrational numbers Q' (which cannot). Every real number has a decimal representation that is either terminating, non-terminating and recurring (both rational), or non-terminating and non-recurring (irrational) -- this decimal test is the fastest way to classify any given number.

This unit covers the real number system and its properties (closure, commutative, associative, identity, inverse, distributive, equality, and order properties), representing irrational numbers geometrically on a number line, the laws of radicals and indices, surds and rationalizing denominators, and practical applications of real numbers in daily life including temperature conversion between Celsius, Fahrenheit, and Kelvin, and profit/loss calculations in business.

Learning Objectives

- Describe the set of real numbers as the union of rational and irrational numbers
- Classify a decimal number as rational or irrational using its terminating/recurring pattern
- Represent irrational numbers on a number line using geometric construction (Pythagoras theorem)
- State and apply the additive, multiplicative, distributive, equality, and order properties of real numbers
- Apply the laws of radicals and indices to simplify radical and exponential expressions
- Identify surds, classify them as monomial or binomial, and rationalize denominators using conjugates
- Apply real number concepts to real-life problems: temperature conversion, profit and loss, ratio division



Key Concepts

1.1 The Real Number System

The set of rational numbers is $Q = \{p/q : p, q \in \mathbb{Z}, q \neq 0\}$, and the set of irrational numbers Q' contains every real number that cannot be written this way. The set of real numbers is their union: $R = Q \cup Q'$. Every rational number's decimal expansion either terminates (e.g. $1/4 = 0.25$) or is non-terminating but recurring (e.g. $1/3 = 0.333... = 0.\bar{3}$); every irrational number's decimal expansion is non-terminating and non-recurring (e.g. $\pi = 3.14159265...$).

This decimal-pattern test is the standard way to classify any given decimal number: check whether it stops, repeats a fixed block forever, or never repeats -- the three cases correspond exactly to terminating rational, recurring rational, and irrational respectively.

1.2 Representing Irrational Numbers on a Number Line

An irrational number like $\sqrt{5}$ can be placed exactly on a number line using geometric construction based on the Pythagoras theorem. Draw $OA = 2$ units along the number line, then $AB = 1$ unit perpendicular to OA at A ; the hypotenuse OB of right triangle OAB satisfies $OB^2 = OA^2 + AB^2 = 4 + 1 = 5$, so $OB = \sqrt{5}$. Swinging an arc of this radius from O back onto the number line marks the exact point representing $\sqrt{5}$.

This method generalizes to any $\sqrt{n}$: choose OA and AB so that $OA^2 + AB^2 = n$, construct the right triangle, then transfer the hypotenuse length to the number line with a compass arc.

1.3 Properties of Real Numbers

Real numbers satisfy additive and multiplicative closure, commutative, associative, identity (0 for addition, 1 for multiplication), and inverse properties (except 0 has no multiplicative inverse), plus the distributive properties linking addition/subtraction and multiplication: $a(b+c) = ab+ac$ and $(a+b)c = ac+bc$, in both left and right forms.



Real numbers also satisfy properties of equality (reflexive, symmetric, transitive, additive, multiplicative, and cancellation) and properties of order (trichotomy, transitive, additive, multiplicative, division, and reciprocal) -- these order properties are essential for solving and justifying steps in inequalities.

1.4 Radicals, Indices, Surds, and Rationalization

For a positive integer $n > 1$ and real number a , $x = \sqrt[n]{a}$ is the n th root of a , with exponential form $x = a^{1/n}$. The laws of radicals ($\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$, $\sqrt[n]{a/b} = \sqrt[n]{a} / \sqrt[n]{b}$, etc.) mirror the laws of indices ($a^m \cdot a^n = a^{m+n}$, $(a^m)^n = a^{mn}$, etc.) and are used together to simplify radical expressions into exponential form and back.

A surd is an irrational radical with a rational radicand, e.g. $\sqrt{5}$ (but not $\sqrt{9}$, which simplifies to the whole number 3). A monomial surd has one term ($\sqrt{5}$); a binomial surd is the sum of two monomial surds ($\sqrt{3} + \sqrt{5}$); $\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$ are conjugate surds, and their product is always rational -- this is the key fact used to rationalize a denominator of the form $a + b\sqrt{x}$ by multiplying numerator and denominator by its conjugate.

1.5 Applications: Temperature Conversion

Real numbers model everyday quantities like temperature, which is measured on three related scales: Kelvin ($K = ^\circ C + 273$), Celsius ($^\circ C = 5/9(F - 32)$), and Fahrenheit ($^\circ F = 9^\circ C/5 + 32$). These formulas let any temperature reading be converted between scales, which is essential in science, engineering, and everyday weather reporting.

1.6 Applications: Profit, Loss, and Ratios

Business calculations rely on real number arithmetic: Profit = Selling Price - Cost Price (with Profit% = $(\text{profit}/CP) \times 100\%$), and Loss = Cost Price - Selling Price (with Loss% = $(\text{loss}/CP) \times 100\%$). When a profit or investment must be split among partners in a given ratio, each share is found by multiplying the total by that partner's fraction of the sum of the ratio terms.

Important Definitions

What is the set of rational numbers Q ?



$Q = \{p/q : p, q \in \mathbb{Z}, q \neq 0\}$ -- all numbers expressible as a ratio of two integers with nonzero denominator.

What is the set of irrational numbers Q' ?

The set of real numbers that cannot be expressed as a quotient of two integers.

How is the set of real numbers R defined?

$R = Q \cup Q'$, the union of the rational and irrational numbers.

What is a terminating decimal number?

A decimal number with a finite number of digits after the decimal point, e.g. $1/4 = 0.25$.

What is a non-terminating, recurring decimal number?

A decimal with an infinitely repeating pattern of digits after the decimal point, e.g. $1/3 = 0.\bar{3}$; such numbers are rational.

What is a surd?

An irrational radical with a rational radicand, e.g. $\sqrt{5}$ ($\sqrt{9}$ is not a surd since it simplifies to the rational number 3).

What are conjugate surds?

$\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$, whose product is always a rational number: $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$.

What is the radical form $x = \sqrt[n]{a}$, and its exponential form?

$x = \sqrt[n]{a}$ means x is the n th root of a ; its exponential form is $x = a^{(1/n)}$.

What is the trichotomy property of order?

For all $a, b \in \mathbb{R}$, exactly one of $a = b$, $a > b$, or $a < b$ holds.

What formula converts Celsius to Fahrenheit?

$^{\circ}\text{F} = (9^{\circ}\text{C}/5) + 32$.

Key Facts and Relations

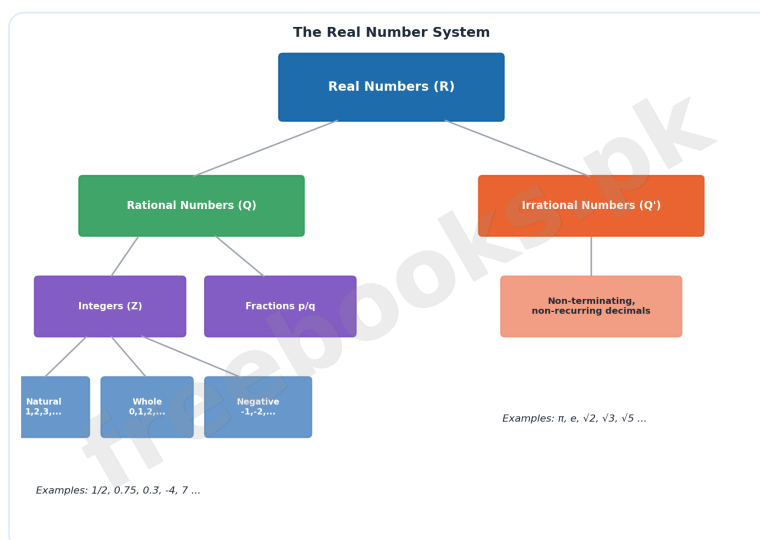
Topic	Key Fact / Relation
Real numbers as a union	$R = Q \cup Q'$
Terminating decimal example	$1/4 = 0.25$



Topic	Key Fact / Relation
Recurring decimal example	$1/3 = 0.3 = 0.333\dots$
Radical to exponential form	$\sqrt[n]{a} = a^{(1/n)}$
Product of conjugate surds	$(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = a - b$
Distributive property (left)	$a(b+c) = ab + ac$
Kelvin from Celsius	$K = ^\circ\text{C} + 273$
Celsius from Fahrenheit	$^\circ\text{C} = (5/9)(F - 32)$
Fahrenheit from Celsius	$^\circ\text{F} = (9^\circ\text{C}/5) + 32$
Profit and loss percentage	Profit% = (Profit/CP)×100%, Loss% = (Loss/CP)×100%

Diagrams

The Real Number System: A tree diagram showing R as the union of Rational and Irrational numbers, with rational numbers further branching into integers and fractions, and integers into natural, whole, and negative numbers

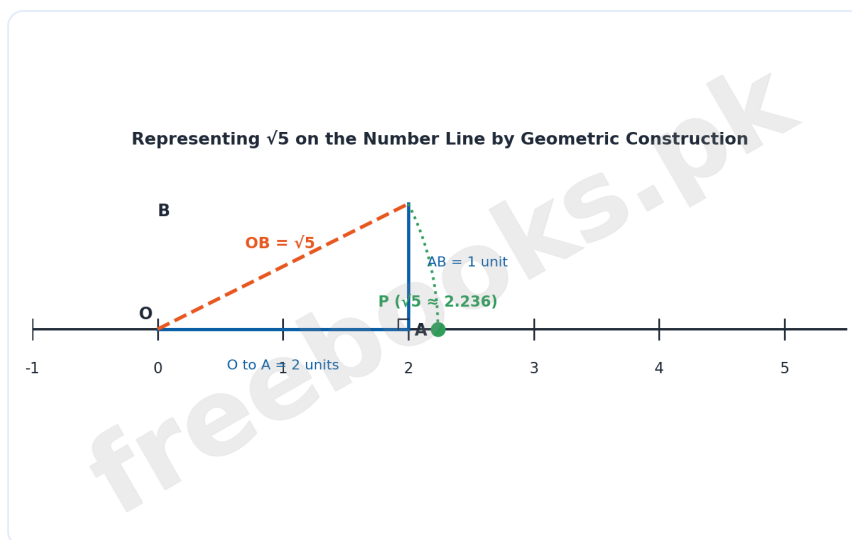


The Real Number System

Representing $\sqrt{5}$ on the Number Line by Geometric Construction: The right-triangle construction OAB with OA=2, AB=1, giving hypotenuse OB= $\sqrt{5}$ by



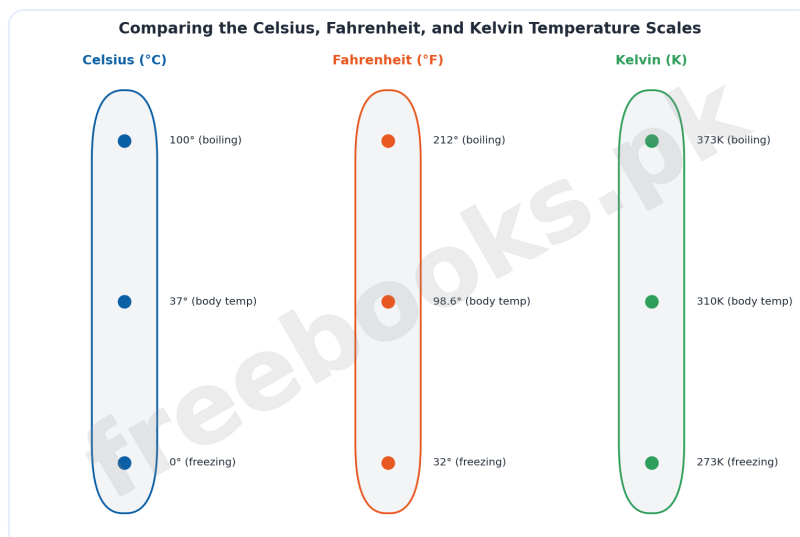
the Pythagoras theorem, then transferred to the number line with a compass arc to mark the exact point $P=\sqrt{5}$



Representing $\sqrt{5}$ on the Number Line by Geometric Construction

Comparing the Celsius, Fahrenheit, and Kelvin Temperature Scales:

Three parallel scales showing the freezing point, human body temperature, and boiling point of water on each of the Celsius, Fahrenheit, and Kelvin scales



Comparing the Celsius, Fahrenheit, and Kelvin Temperature Scales



Solved Examples

Example 1: Classifying Decimal Numbers as Rational or Irrational

Problem: Identify the following as rational or irrational: (i) 0.35 (ii) 0.444... (iii) 3.36788542...

1. 0.35 has a finite number of digits after the decimal point, so it is a terminating decimal -- therefore rational.
2. 0.444... repeats the digit 4 forever in a fixed pattern, so it is a non-terminating, recurring decimal -- therefore rational.
3. 3.36788542... shows no repeating pattern and continues indefinitely, so it is non-terminating and non-recurring -- therefore irrational.
4. Compare each case against the decimal-pattern test: terminates → rational; repeats a block forever → rational; never repeats → irrational.
5. Final answer: 0.35 is rational, 0.444... is rational, 3.36788542... is irrational.

Example 2: Representing $\sqrt{5}$ on a Number Line

Problem: Represent $\sqrt{5}$ on a number line using geometric construction.

1. Draw $OA=2$ units along the number line from the origin O .
2. At point A , draw $AB=1$ unit perpendicular to OA , forming right triangle OAB .
3. Apply the Pythagoras theorem: $(OB)^2 = (OA)^2 + (AB)^2 = 2^2 + 1^2 = 4 + 1 = 5$, so $OB = \sqrt{5}$.
4. With centre O and radius $OB = \sqrt{5}$, draw an arc that meets the number line at point P .
5. Final answer: point P represents $\sqrt{5}$ on the number line, since $|OP| = OB = \sqrt{5}$.



Example 3: Expressing a Recurring Decimal as p/q

Problem: Express $0.\bar{93}$ (i.e. $0.939393\dots$) as a rational number p/q .

1. Let $x=0.939393\dots$, so $100x=93.939393\dots$ (multiplying by 100 since the repeating block has 2 digits).
2. Subtract the original equation from the multiplied one: $100x - x = 93.939393\dots - 0.939393\dots$
3. This gives $99x=93$.
4. Solve for x : $x=93/99$.
5. Final answer: $0.\bar{93}=93/99$, expressed in the form p/q with integers p and q .

Example 4: Simplifying a Radical Expression

Problem: Simplify $\sqrt[n]{16x^4y^8}$ for $n=4$, i.e. the 4th root of $16x^4y^8$.

1. Rewrite the radical in exponential form: $(16x^4y^8)^{1/4}$, using $\sqrt[n]{a}=a^{1/n}$.
2. Apply the power-of-a-product law: $=(16)^{1/4}(x^4)^{1/4}(y^8)^{1/4}$.
3. Simplify each factor: $16^{1/4}=2^{(4 \times 1/4)}=2$; $(x^4)^{1/4}=x^{(4 \times 1/4)}=x$; $(y^8)^{1/4}=y^{(8 \times 1/4)}=y^2$.
4. Combine the simplified factors together.
5. Final answer: $\sqrt[4]{16x^4y^8} = 2xy^2$.

Example 5: Rationalizing a Denominator

Problem: Rationalize the denominator of $3/(\sqrt{5}+\sqrt{2})$.

1. Multiply numerator and denominator by the conjugate of the denominator, $\sqrt{5}-\sqrt{2}$.
2. This gives $3(\sqrt{5}-\sqrt{2}) / [(\sqrt{5}+\sqrt{2})(\sqrt{5}-\sqrt{2})]$.
3. Simplify the denominator using the conjugate product rule: $(\sqrt{5})^2 - (\sqrt{2})^2 = 5 - 2 = 3$.



4. The expression becomes $3(\sqrt{5}-\sqrt{2})/3$.
5. Final answer: the 3's cancel, leaving $\sqrt{5}-\sqrt{2}$ as the rationalized form.

Example 6: Verifying the Distributive Property

Problem: If $a=2/3$, $b=3/2$, $c=5/3$, verify the left distributive property $a(b+c)=ab+ac$.

1. Compute the LHS: $a(b+c) = (2/3)(3/2+5/3) = (2/3)(9/6+10/6) = (2/3)(19/6) = 19/9$.
2. Compute ab : $(2/3)(3/2) = 1$.
3. Compute ac : $(2/3)(5/3) = 10/9$.
4. Add $ab+ac$: $1 + 10/9 = 9/9 + 10/9 = 19/9$.
5. Final answer: $LHS=RHS=19/9$, so $a(b+c)=ab+ac$ is verified.

Example 7: Applying Temperature Conversion

Problem: Normal human body temperature is 98.6°F . Convert it into Celsius and Kelvin.

1. Convert Fahrenheit to Celsius using $^{\circ}\text{C}=(5/9)(\text{F}-32)$: $^{\circ}\text{C}=(5/9)(98.6-32)=(5/9)(66.6)$.
2. Compute: $(5/9)(66.6) = 37$.
3. So the Celsius reading is 37°C .
4. Convert Celsius to Kelvin using $\text{K}=^{\circ}\text{C}+273$: $\text{K}=37+273=310$.
5. Final answer: $98.6^{\circ}\text{F} = 37^{\circ}\text{C} = 310 \text{ K}$.

Example 8: Applying Profit Percentage

Problem: Hamail purchased a bicycle for Rs. 6590 and sold it for Rs. 6850. Find the profit percentage.

1. Identify $\text{CP}=\text{Rs. } 6590$ and $\text{SP}=\text{Rs. } 6850$.



2. Compute profit: Profit = SP - CP = 6850 - 6590 = Rs. 260.
3. Apply the profit percentage formula: Profit% = (Profit/CP) $\times$ 100%.
4. Substitute: $(260/6590) \times 100\% \approx 3.94\%$.
5. Final answer: the profit percentage is approximately 4%.

Short Questions & Answers

How is the set of real numbers R related to Q and Q'?

R is the union of the rational numbers Q and the irrational numbers Q': $R = Q \cup Q'$.

How can you tell if a decimal number is rational or irrational just by looking at its digits?

If it terminates or repeats a fixed block of digits forever, it is rational; if it never terminates and never repeats, it is irrational.

What theorem is used to represent an irrational number like $\sqrt{5}$ on a number line?

The Pythagoras theorem, applied to a right triangle whose hypotenuse has the required irrational length.

What makes $\sqrt{9}$ not a surd, even though it involves a radical sign?

Because $\sqrt{9}$ simplifies to the rational number 3, and a surd must be irrational; only radicals that stay irrational after simplification count as surds.

Why does rationalizing a denominator with a conjugate always work?

Because the product of conjugate surds $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$ is always rational, eliminating the radical from the denominator.

Which real number has no multiplicative inverse, and why?

0, because there is no real number x such that $0 \times x = 1$.

What is the formula for converting Celsius to Kelvin?

$K = ^\circ\text{C} + 273$.



Long Questions & Answers

Explain how a decimal number's pattern is used to classify it as rational or irrational, and describe how an irrational number can be represented exactly on a number line.

What are the three possible decimal patterns a real number can have?

A real number's decimal expansion either terminates after finitely many digits, repeats a fixed block of digits forever (non-terminating but recurring), or continues forever without ever repeating a fixed block (non-terminating and non-recurring).

Which of these patterns correspond to rational numbers, and which to irrational?

Terminating decimals and non-terminating recurring decimals are both rational, since both can be converted into the form p/q with integers p and q ; non-terminating, non-recurring decimals are irrational, since no such fraction exists for them.

How is an irrational number like $\sqrt{5}$ placed exactly on a number line?

By geometric construction: build a right triangle with legs chosen so their squares sum to 5 (e.g. $OA=2$, $AB=1$), apply the Pythagoras theorem to get hypotenuse $OB=\sqrt{5}$, then draw a compass arc of that radius from O to mark the exact point on the number line.

Why can't this construction method be replaced by just estimating the decimal value and marking that point?

Because a decimal estimate like 2.236 is only an approximation of $\sqrt{5}$, not the exact value; the geometric construction places the point at the true, exact irrational distance from the origin, using the guaranteed exactness of the Pythagoras theorem rather than a rounded decimal guess.



Explain the laws of radicals and indices, what a surd is, and how the denominator of a fraction containing a surd is rationalized.

What is the relationship between radical form and exponential form?

The radical $x = \sqrt[n]{a}$ is equivalent to the exponential form $x = a^{(1/n)}$; this equivalence lets every law of radicals (like $\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}$) be matched to a corresponding law of indices (like $(ab)^n = a^n b^n$), so radical expressions can be simplified by converting to exponents and applying index laws.

What distinguishes a surd from any other irrational radical?

A surd is specifically an irrational radical with a RATIONAL radicand -- so $\sqrt{5}$ is a surd because 5 is rational and $\sqrt{5}$ is irrational, but $\sqrt{\pi}$ is not a surd even though it is irrational, because π itself is not rational.

What are conjugate surds, and what special property do they have?

$\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$ are conjugate surds of each other; their product $(\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b}) = a - b$ is always a rational number, since the radical terms cancel out through the difference-of-squares pattern.

How is this conjugate property used to rationalize a denominator like $3/(\sqrt{5} + \sqrt{2})$?

Multiply both the numerator and denominator by the conjugate of the denominator, $\sqrt{5} - \sqrt{2}$; the denominator becomes $(\sqrt{5} + \sqrt{2})(\sqrt{5} - \sqrt{2}) = 5 - 2 = 3$, a rational number, while the numerator becomes $3(\sqrt{5} - \sqrt{2})$, so the fraction simplifies to $\sqrt{5} - \sqrt{2}$ with no radical left in the denominator.

Multiple Choice Questions (MCQs)

$\sqrt{7}$ is:

- A. An integer
- B. A rational number
- C. An irrational number
- D. A natural number

Answer: C. An irrational number



Explanation: 7 is not a perfect square, so $\sqrt{7}$ has a non-terminating, non-recurring decimal expansion, making it irrational.

The set of real numbers R is defined as:

- A. Q only
- B. Q' only
- C. $Q \cup Q'$
- D. $Q \cap Q'$

Answer: C. $Q \cup Q'$

Explanation: R is the union of the rational numbers Q and the irrational numbers Q' .

A terminating decimal number is always:

- A. Irrational
- B. Rational
- C. A surd
- D. Undefined

Answer: B. Rational

Explanation: Terminating decimals can always be written as p/q , so they are rational.

Which of these is NOT a surd?

- A. $\sqrt{5}$
- B. $\sqrt{7}$
- C. $\sqrt{9}$
- D. $\sqrt{11}$

Answer: C. $\sqrt{9}$

Explanation: $\sqrt{9}$ simplifies to the rational number 3, so it fails the requirement that a surd be irrational.

The product of the conjugate surds $(\sqrt{a}+\sqrt{b})$ and $(\sqrt{a}-\sqrt{b})$ is:

- A. Always irrational
- B. Always negative
- C. Always rational (a-b)
- D. Always zero

Answer: C. Always rational (a-b)

Explanation: By the difference-of-squares pattern, $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=a-b$, which is rational whenever a and b are rational.



0 has no multiplicative inverse because:

- A. 0 is not a real number
- B. No real number x satisfies $0 \times x = 1$
- C. 0 is irrational
- D. Multiplication is not defined for 0

Answer: B. No real number x satisfies $0 \times x = 1$

Explanation: There is no real x with $0 \times x = 1$, so 0 has no multiplicative inverse, unlike every other real number.

The formula to convert Celsius to Fahrenheit is:

- A. $^{\circ}\text{F} = ^{\circ}\text{C} + 32$
- B. $^{\circ}\text{F} = (9^{\circ}\text{C}/5) + 32$
- C. $^{\circ}\text{F} = (5^{\circ}\text{C}/9) - 32$
- D. $^{\circ}\text{F} = ^{\circ}\text{C} \times 32$

Answer: B. $^{\circ}\text{F} = (9^{\circ}\text{C}/5) + 32$

Explanation: The correct conversion formula is $^{\circ}\text{F} = (9^{\circ}\text{C}/5) + 32$.

If Profit = SP - CP, then Profit% is calculated as:

- A. $(\text{Profit}/\text{SP}) \times 100\%$
- B. $(\text{Profit}/\text{CP}) \times 100\%$
- C. $(\text{CP}/\text{Profit}) \times 100\%$
- D. $(\text{SP}/\text{CP}) \times 100\%$

Answer: B. $(\text{Profit}/\text{CP}) \times 100\%$

Explanation: Profit percentage is always calculated relative to the Cost Price: $(\text{Profit}/\text{CP}) \times 100\%$.

The trichotomy property states that for any $a, b \in \mathbb{R}$:

- A. $a=b$ always
- B. Exactly one of $a=b$, $a>b$, or $a<b$ holds
- C. $a>b$ always
- D. a and b must be equal or irrational

Answer: B. Exactly one of $a=b$, $a>b$, or $a<b$ holds

Explanation: Trichotomy guarantees exactly one of the three relations ($=$, $>$, $<$) holds between any two real numbers.

Which right-triangle construction is used to represent $\sqrt{5}$ on a number line?



- A. Legs of 1 and 1
- B. Legs of 2 and 1
- C. Legs of 3 and 1
- D. Legs of 5 and 0

Answer: B. Legs of 2 and 1

Explanation: With legs $OA=2$ and $AB=1$, the Pythagoras theorem gives hypotenuse $OB=\sqrt{4+1}=\sqrt{5}$.

Quick Revision Summary

- Real numbers: $R = Q \cup Q'$ (rational numbers union irrational numbers)
- Terminating or recurring decimals = rational; non-terminating non-recurring decimals = irrational
- Irrational numbers are placed exactly on a number line using Pythagoras-theorem geometric construction
- Real number properties: closure, commutative, associative, identity, inverse (additive & multiplicative), distributive
- Properties of equality: reflexive, symmetric, transitive, additive, multiplicative, cancellation
- Properties of order: trichotomy, transitive, additive, multiplicative, division, reciprocal
- Radical form $\sqrt[n]{a} = \text{exponential form } a^{(1/n)}$; laws of radicals mirror laws of indices
- Surd = irrational radical with rational radicand; conjugate surds' product $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=a-b$ is always rational
- Rationalize a denominator by multiplying numerator and denominator by the denominator's conjugate
- Applications: $K=^{\circ}\text{C}+273$, $^{\circ}\text{C}=(5/9)(F-32)$, $^{\circ}\text{F}=(9^{\circ}\text{C}/5)+32$; $\text{Profit}\%=(\text{Profit}/\text{CP})\times 100\%$, $\text{Loss}\%=(\text{Loss}/\text{CP})\times 100\%$

Exam Tips

- When classifying a decimal as rational or irrational, look specifically for a REPEATING BLOCK of digits, not just any decimal digits -- $0.121121112\dots$ looks repetitive but never repeats a fixed block, so it is irrational



- For number-line construction problems, always identify the two leg lengths whose squares sum to the target number under the radical before drawing anything
- Memorize which properties apply to equality ($=$) versus order ($<$, $>$) separately -- multiplying an inequality by a negative number flips the direction, but this never happens with equalities
- When simplifying radicals, convert to exponential form first ($\sqrt[n]{a} = a^{(1/n)}$) -- it's much easier to apply index laws than to juggle radical notation directly
- Always double-check which surd is the correct conjugate: it's the SAME two terms with only the middle sign flipped, e.g. the conjugate of $\sqrt{5}-\sqrt{3}$ is $\sqrt{5}+\sqrt{3}$, not $-\sqrt{5}-\sqrt{3}$
- For profit/loss and ratio-division word problems, write out exactly what is given (CP, SP, ratio, total) before choosing a formula -- misreading which value is the total versus a part is the most common error



Unit 2: Logarithms

Scientific notation writes very large or very small numbers compactly as $a \times 10^n$, where $1 \leq a < 10$ and n is an integer -- essential in science and engineering for handling numbers like the speed of light or the size of an atom. Logarithms extend this idea: the logarithm of a number x to base b , written $\log_b(x)=y$, answers the question 'to what power y must b be raised to get x ?', and is defined equivalently by $b^y=x$. Introduced by John Napier and extended by Henry Briggs (who developed the common logarithm and logarithm table), logarithms turn multiplication into addition and division into subtraction, making them powerful tools for simplifying complex calculations.

This unit covers converting numbers between ordinary and scientific notation, the logarithmic-exponential relationship, common logarithms (base 10) including the characteristic and mantissa and how to use logarithm/antilogarithm tables, natural logarithms (base e), and the four laws of logarithm (product, quotient, power, change of base) with their proofs -- closing with real-world applications like the decibel sound scale, earthquake magnitude, compound growth, and population modelling.

Learning Objectives

- Express a number in scientific notation and convert scientific notation back to ordinary notation
- Describe the logarithm of a number and convert between logarithmic and exponential form
- Differentiate between common logarithm (base 10) and natural logarithm (base e)
- Find the characteristic and mantissa of a common logarithm, and use logarithm/antilogarithm tables
- State and prove the product, quotient, power, and change-of-base laws of logarithm
- Apply the laws of logarithm to expand, combine, and simplify logarithmic expressions and solve logarithmic equations



- Apply logarithms to real-life problems: decibel sound levels, earthquake magnitude, and exponential growth/decay

Key Concepts

2.1 Scientific Notation

A number in scientific notation is written $a \times 10^n$, where $1 \leq a < 10$ and $n \in \mathbb{Z}$. To convert an ordinary number to scientific notation, move the decimal point until only one non-zero digit remains before it, and count the number of places moved -- this count becomes n (positive if the original number was ≥ 10 , negative if it was < 1). To convert back, move the decimal point $|n|$ places right (if n is positive) or left (if n is negative).

2.2 Logarithm of a Real Number

The logarithm of x to base b , $\log_b(x)=y$, means $b^y=x$, where $b>0$, $x>0$, and $b \neq 1$. Here b is the base, x is the result whose logarithm is being taken, and y is the exponent (the logarithm itself). Converting between forms is direct:

$\log_b(x)=y \iff b^y=x$, e.g. $\log_2 8=3 \iff 2^3=8$.

2.3 Common Logarithm: Characteristic and Mantissa

The common logarithm uses base 10, written $\log_{10}$ or simply $\log$. Every common logarithm splits into a characteristic (the integer part, found from the number of digits) and a mantissa (the decimal part, always positive, found from a logarithm table): $\log(\text{Number}) = \text{Characteristic} + \text{Mantissa}$. For numbers ≥ 1 , $\text{characteristic} = (\text{digits before decimal point}) - 1$; for numbers < 1 , $\text{characteristic} = -(\text{zeros between decimal point and first non-zero digit} + 1)$, written with a bar (e.g. $\bar{2}$).

Antilogarithm is the inverse operation: given a logarithm value, find the original number using an antilogarithm table, splitting the value into characteristic and (positive) mantissa, looking up the mantissa's table value, then placing the decimal point according to the characteristic.



2.4 Natural Logarithm

The natural logarithm uses base $e \approx 2.71828$ (introduced by Leonhard Euler), written $\ln(x)$. It is used extensively in calculus and to model exponential growth and decay. Common logarithm (base 10) is used in everyday scientific/engineering calculations; natural logarithm (base e) is used in higher mathematics and growth/decay modelling.

2.5 Laws of Logarithm

The four laws of logarithm are: Product Law, $\log_b(xy) = \log_b(x) + \log_b(y)$; Quotient Law, $\log_b(x/y) = \log_b(x) - \log_b(y)$; Power Law, $\log_b(x^n) = n \cdot \log_b(x)$; and Change of Base Law, $\log_b(x) = \log_a(x) / \log_a(b)$. Each is proved by writing the logarithms in exponential form, combining the exponents using the corresponding index law, then converting back to logarithmic form.

These laws let complex logarithmic expressions be expanded into sums/differences of simpler logarithms, or combined from several logarithms into a single one -- both directions are used constantly when simplifying expressions or solving logarithmic equations.

2.6 Applications of Logarithm

Logarithms model real-world phenomena that span huge ranges of scale: the decibel sound scale $L = 40 \log_{10}(I/I_0)$ measures sound intensity relative to a reference; the earthquake magnitude scale $M = \log_{10}(A/A_0)$ measures amplitude relative to a reference; and exponential growth models like $y = P(1+r)^t$ (compound interest, population growth) are solved for the unknown time t by taking logarithms of both sides.

Important Definitions

What is scientific notation?

A way of writing a number as $a \times 10^n$, where $1 \leq a < 10$ and n is an integer.

What is the logarithm of x to base b ?

The exponent y such that $b^y = x$, written $\log_b(x) = y$, where $b > 0$, $x > 0$, $b \neq 1$.

What is a common logarithm?



A logarithm with base 10, written $\log_{10}$ or simply $\log$.

What is a natural logarithm?

A logarithm with base $e \approx 2.71828$, written $\ln$.

What is the characteristic of a common logarithm?

The integer part of the logarithm, found from the number of digits before the decimal point (for numbers ≥ 1) or the number of leading zeros after the decimal point (for numbers < 1).

What is the mantissa of a common logarithm?

The decimal part of the logarithm, always positive, found using a logarithm table.

What is an antilogarithm?

The inverse operation of a logarithm: given $y = \log_b(x)$, the antilogarithm finds x .

State the Product Law of logarithm.

$$\log_b(xy) = \log_b(x) + \log_b(y).$$

State the Power Law of logarithm.

$$\log_b(x^n) = n \cdot \log_b(x).$$

State the Change of Base Law of logarithm.

$$\log_b(x) = \log_a(x) / \log_a(b), \text{ converting from base } b \text{ to any base } a.$$

Key Facts and Relations

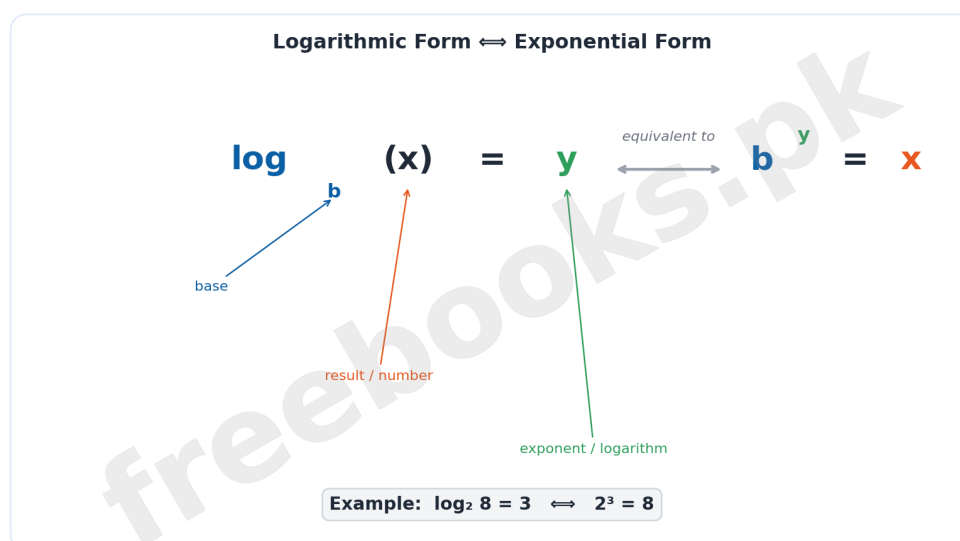
Topic	Key Fact / Relation
Scientific notation form	$a \times 10^n$, where $1 \leq a < 10$, $n \in \mathbb{Z}$
Logarithmic-exponential equivalence	$\log_b(x) = y \iff b^y = x$
Common logarithm decomposition	$\log(\text{Number}) = \text{Characteristic} + \text{Mantissa}$
Characteristic for a number ≥ 1	Characteristic = (digits before decimal point) - 1
Characteristic for a number < 1	Characteristic = -(zeros between decimal point and first non-zero digit + 1)
Product Law	$\log_b(xy) = \log_b(x) + \log_b(y)$



Topic	Key Fact / Relation
Quotient Law	$\log_b(x/y) = \log_b(x) - \log_b(y)$
Power Law	$\log_b(x^n) = n \cdot \log_b(x)$
Change of Base Law	$\log_b(x) = \log_a(x) / \log_a(b)$
Decibel and earthquake magnitude formulas	$L = 40 \log_{10}(I/I_0); M = \log_{10}(A/A_0)$

Diagrams

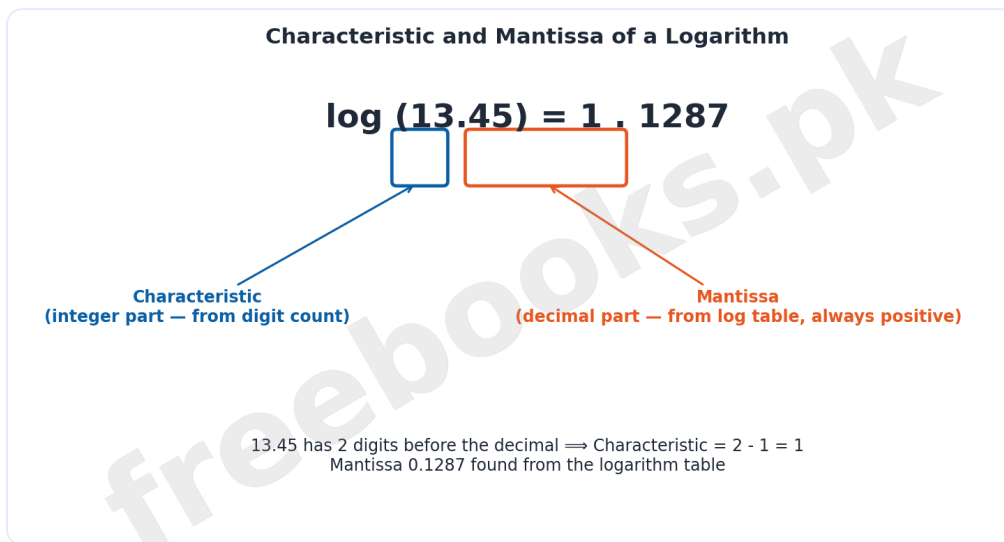
Logarithmic Form $\Leftrightarrow$ Exponential Form: A labelled diagram showing the equivalence between $\log_b(x)=y$ and $b^y=x$, with the base, result, and exponent each labelled, illustrated with the example $\log_2 8=3 \Leftrightarrow 2^3=8$



Logarithmic Form $\Leftrightarrow$ Exponential Form

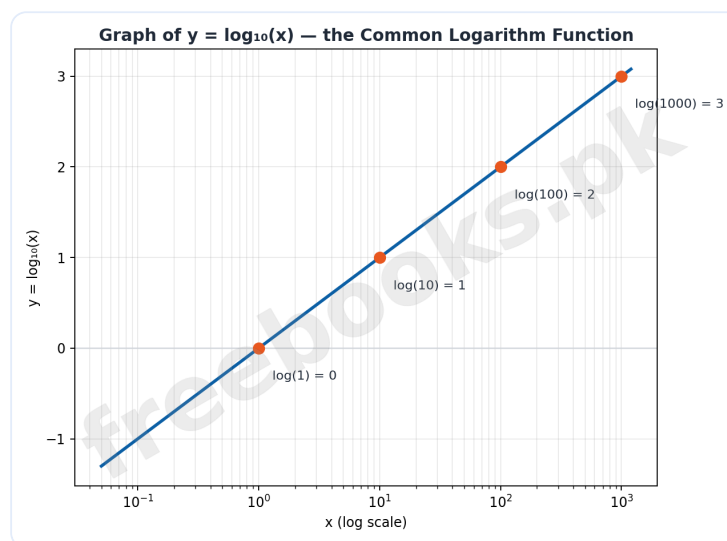
Characteristic and Mantissa of a Logarithm: A labelled breakdown of $\log(13.45)=1.1287$ showing the characteristic (integer part, from digit count) and the mantissa (decimal part, from the logarithm table)





Characteristic and Mantissa of a Logarithm

Graph of $y = \log_{10}(x)$ — the Common Logarithm Function: The common logarithm function plotted on a logarithmic x-axis, with key points $\log(1)=0$, $\log(10)=1$, $\log(100)=2$, and $\log(1000)=3$ marked



Graph of $y = \log_{10}(x)$ — the Common Logarithm Function

Solved Examples

Example 1: Converting to Scientific Notation

Problem: Convert 78,000,000 to scientific notation.

1. Move the decimal point left until only one non-zero digit remains before it:
7.8.



2. Count the number of places the decimal point moved: 7 places.
3. Since the decimal moved left (the original number was greater than 1), the exponent is positive.
4. Write the number as coefficient $\times 10^{(\text{places moved})}$.
5. Final answer: $78,000,000 = 7.8 \times 10^7$.

Example 2: Converting Logarithmic and Exponential Forms

Problem: Convert $\log_2 8 = 3$ to exponential form, and find x if $\log_5 25 = x$.

1. For $\log_2 8 = 3$: using $\log_b(x) = y \iff b^y = x$ with $b=2$, $x=8$, $y=3$, the exponential form is $2^3 = 8$.
2. For $\log_5 25 = x$: write in exponential form, $5^x = 25$.
3. Rewrite 25 as a power of 5: $5^x = 5^2$.
4. Since the bases match, equate the exponents: $x=2$.
5. Final answer: $\log_2 8 = 3$ becomes $2^3 = 8$, and $\log_5 25 = x$ gives $x=2$.

Example 3: Finding the Characteristic and Mantissa

Problem: Find the characteristic of $\log 725$ and $\log 0.00045$.

1. For $\log 725$: this number has 3 digits before the decimal point, so characteristic $= 3 - 1 = 2$.
2. For $\log 0.00045$: there are 3 zeros between the decimal point and the first non-zero digit (4), so characteristic $= -(3 + 1) = -4$, written $\bar{4}$.
3. Cross-check using scientific notation: $725 = 7.25 \times 10^2$, confirming characteristic 2; $0.00045 = 4.5 \times 10^{-4}$, confirming characteristic -4.
4. This scientific-notation cross-check works because the exponent of 10 always equals the characteristic.
5. Final answer: characteristic of $\log 725$ is 2; characteristic of $\log 0.00045$ is $\bar{4}$ (-4).



Example 4: Finding a Common Logarithm Using Tables

Problem: Find $\log 13.45$ using the logarithm table.

1. Separate the integral and decimal parts: integral part 13, decimal part 45.
2. Find the characteristic: number of digits before the decimal point (2) minus 1, giving characteristic=1.
3. Look up the logarithm table at row 13, column 4 (from the digit '4' in 13.45): table value 1271.
4. Add the mean difference for the next digit (5): row 13, mean-difference column 5, value 16; so $1271+16=1287$, giving mantissa 0.1287.
5. Final answer: $\log 13.45 = \text{characteristic} + \text{mantissa} = 1 + 0.1287 = 1.1287$.

Example 5: Finding an Antilogarithm Using Tables

Problem: Find the antilogarithm of 2.1245.

1. Separate the characteristic and mantissa: characteristic=2, mantissa=0.1245.
2. Look up the antilogarithm table at row .12, column 4: table value 1330.
3. Add the mean difference at row .12, column 5: value 2; so $1330+2=1332$.
4. Since the characteristic is 2, place the decimal point 2 digits to the right of the reference position, giving 133.2.
5. Final answer: $\text{antilog}(2.1245) \approx 133.2$.

Example 6: Proving and Applying the Product Law

Problem: Prove $\log_b(xy) = \log_b(x) + \log_b(y)$, then use it to expand $\log_3(20)$.

1. Let $m = \log_b(x)$ and $n = \log_b(y)$, so in exponential form $x = b^m$ and $y = b^n$.



2. Multiply: $xy = b^m \cdot b^n = b^{(m+n)}$, so in logarithmic form $\log_b(xy) = m+n = \log_b(x) + \log_b(y)$, proving the law.
3. Apply to $\log_3(20)$: rewrite $20 = 2^2 \times 5$, so $\log_3(20) = \log_3(2^2) + \log_3(5)$.
4. Apply the Power Law to the first term: $\log_3(2^2) = 2\log_3 2$.
5. Final answer: $\log_3(20) = 2\log_3 2 + \log_3 5$.

Example 7: Combining Logarithms into a Single Logarithm

Problem: Write $2\log_3 10 - \log_3 4$ as a single logarithm.

1. Apply the Power Law to the first term: $2\log_3 10 = \log_3(10^2) = \log_3 100$.
2. Rewrite the expression as $\log_3 100 - \log_3 4$.
3. Apply the Quotient Law: $\log_3 100 - \log_3 4 = \log_3(100/4)$.
4. Simplify the fraction: $100/4 = 25$.
5. Final answer: $2\log_3 10 - \log_3 4 = \log_3 25$.

Example 8: Real-Life Application: Decibel Sound Scale

Problem: The decibel scale is $L = 40\log_{10}(I/I_0)$. If sound intensity I is 10^6 times the reference intensity I_0 , find the sound level L .

1. Substitute $I = 10^6 I_0$ into the formula: $L = 40\log_{10}(10^6 I_0/I_0)$.
2. Cancel I_0 in the fraction: $L = 40\log_{10}(10^6)$.
3. Apply the Power Law: $\log_{10}(10^6) = 6\log_{10}(10) = 6 \times 1 = 6$ (since $\log_{10} 10 = 1$).
4. Multiply: $L = 40 \times 6$.
5. Final answer: $L = 240$ decibels.

Short Questions & Answers

What is the general form of a number written in scientific notation?

$a \times 10^n$, where $1 \leq a < 10$ and n is an integer.

What does $\log_b(x) = y$ mean in exponential form?



$b^y = x$, where $b > 0$, $x > 0$, and $b \neq 1$.

What are the two parts of a common logarithm called?

The characteristic (integer part) and the mantissa (decimal part, always positive).

Which base does a natural logarithm use, and who introduced this constant?

Base $e \approx 2.71828$, a constant introduced by Leonhard Euler.

What does the Quotient Law of logarithm state?

$\log_b(x/y) = \log_b(x) - \log_b(y)$: the logarithm of a quotient is the difference of the logarithms.

Why is $\log(0)$ undefined?

Because there is no real exponent y such that $10^y = 0$; as $y \rightarrow -\infty$, 10^y approaches 0 but never reaches it.

What is the value of $\log(1)$, and why?

0, because $10^0 = 1$ for any nonzero base.

Long Questions & Answers

Explain what the characteristic and mantissa of a common logarithm are, how each is found, and how they combine to give the logarithm of a number.

What is the characteristic of a common logarithm, and how is it found for a number greater than 1?

The characteristic is the integer part of the logarithm; for a number greater than 1, it equals the number of digits to the left of the decimal point minus 1 -- for example, 725 has 3 digits before the decimal, so its characteristic is $3-1=2$.

How is the characteristic found for a number less than 1?

It equals negative the count of zeros between the decimal point and the first non-zero digit, plus 1 -- for example, 0.00045 has 3 such zeros, so the characteristic is $-(3+1)=-4$, written with a bar as $\bar{4}$.



What is the mantissa, and how is it obtained?

The mantissa is the decimal part of the logarithm, always positive regardless of the sign of the characteristic; it is found by looking up the number's significant digits in a logarithm table, combining the table value with the mean-difference correction for the next digit.

How do the characteristic and mantissa combine to give the final logarithm value?

They are simply added together: $\log(\text{Number}) = \text{Characteristic} + \text{Mantissa}$ -- for example, $\log(13.45)$ has characteristic 1 and mantissa 0.1287, giving $\log(13.45)=1.1287$.

Explain the four laws of logarithm, how each is proved from the definition of a logarithm, and how they are used together to simplify expressions.

How is the Product Law $\log_b(xy)=\log_b(x)+\log_b(y)$ proved?

Let $m=\log_b(x)$ and $n=\log_b(y)$, so $x=b^m$ and $y=b^n$ in exponential form; multiplying gives $xy=b^m \cdot b^n=b^{(m+n)}$ by the index law for multiplying powers with the same base, and converting back to logarithmic form gives $\log_b(xy)=m+n=\log_b(x)+\log_b(y)$.

How is the Power Law $\log_b(x^n)=n \cdot \log_b(x)$ proved similarly?

Let $m=\log_b(x)$, so $x=b^m$; raising both sides to the power n gives $x^n=(b^m)^n=b^{(mn)}$ by the index law for a power of a power, and converting back to logarithmic form gives $\log_b(x^n)=mn=n \cdot \log_b(x)$.

What is the Change of Base Law used for, and how is it proved?

It converts a logarithm from base b to any other base a : $\log_b(x)=\log_a(x)/\log_a(b)$; it is proved by taking $\log$ base a of both sides of the exponential form $b^m=x$ (where $m=\log_b(x)$), giving $m \cdot \log_a(b)=\log_a(x)$, then solving for m .



How are these laws combined when expanding or simplifying a real logarithmic expression?

The Product and Quotient Laws break a logarithm of a product or quotient into a sum or difference of simpler logarithms, the Power Law pulls exponents out front as coefficients, and these steps can be reversed to combine several logarithm terms back into a single logarithm -- for example, $2\log_3 10 - \log_3 4$ becomes $\log_3 100 - \log_3 4$ via the Power Law, then $\log_3(100/4) = \log_3 25$ via the Quotient Law.

Multiple Choice Questions (MCQs)

The standard form of 5.2×10^6 is:

- A. 52,000
- B. 520,000
- C. 5,200,000
- D. 52,000,000

Answer: C. 5,200,000

Explanation: Moving the decimal point 6 places to the right gives 5,200,000.

The base of a common logarithm is:

- A. 2
- B. 10
- C. 5
- D. e

Answer: B. 10

Explanation: Common logarithm always uses base 10, written log or $\log_{10}$.

$\log_2 2^3$ equals:

- A. 1
- B. 2
- C. 5
- D. 3

Answer: D. 3

Explanation: By definition, $\log_b(b^n) = n$, so $\log_2 2^3 = 3$.

log 100 equals:

- A. 2
- B. 3



C. 10

D. 1

Answer: A. 2

Explanation: Since $10^2=100$, $\log 100 = 2$.

If $\log 2 = 0.3010$, then $\log 200$ is:

A. 1.3010

B. 0.6010

C. 2.3010

D. 2.6010

Answer: C. 2.3010

Explanation: $\log 200 = \log(2 \times 100) = \log 2 + \log 100 = 0.3010 + 2 = 2.3010$.

$\log(0)$ is:

A. Positive

B. Negative

C. Zero

D. Undefined

Answer: D. Undefined

Explanation: There is no real exponent that makes 10^y equal exactly 0, so $\log(0)$ is undefined.

$\log 5 + \log 3$ equals:

A. $\log 0$

B. $\log 2$

C. $\log(5/3)$

D. $\log 15$

Answer: D. $\log 15$

Explanation: By the Product Law, $\log 5 + \log 3 = \log(5 \times 3) = \log 15$.

$3^4=81$ in logarithmic form is:

A. $\log_3 4=81$

B. $\log_4 3=81$

C. $\log_3 81=4$

D. $\log_4 81=3$

Answer: C. $\log_3 81=4$

Explanation: Using $\log_b(x)=y \iff b^y=x$ with $b=3$, $x=81$, $y=4$ gives $\log_3 81=4$.



The mantissa of a common logarithm is always:

- A. Negative
- B. Zero
- C. Positive
- D. Equal to the characteristic

Answer: C. Positive

Explanation: The mantissa, found from the logarithm table, is by convention always kept positive.

The natural logarithm $\ln(x)$ uses which base?

- A. 10
- B. 2
- C. e
- D. 1

Answer: C. e

Explanation: The natural logarithm uses base $e \approx 2.71828$, the constant introduced by Euler.

Quick Revision Summary

- Scientific notation: $a \times 10^n$, $1 \leq a < 10$, $n \in \mathbb{Z}$; move the decimal point and count places to find n
- Logarithm definition: $\log_b(x) = y \iff b^y = x$, for $b > 0$, $x > 0$, $b \neq 1$
- Common logarithm uses base 10 ($\log$ or $\log_{10}$); natural logarithm uses base $e \approx 2.71828$ ($\ln$)
- $\log(\text{Number}) = \text{Characteristic} + \text{Mantissa}$; characteristic from digit count, mantissa from the log table (always positive)
- Antilogarithm is the inverse of a logarithm, found using an antilogarithm table
- Product Law: $\log_b(xy) = \log_b(x) + \log_b(y)$ | Quotient Law: $\log_b(x/y) = \log_b(x) - \log_b(y)$
- Power Law: $\log_b(x^n) = n \cdot \log_b(x)$ | Change of Base Law: $\log_b(x) = \log_a(x) / \log_a(b)$
- $\log(1) = 0$ for any base; $\log(0)$ is undefined
- Applications: decibel scale $L = 40 \log_{10}(I/I_0)$, earthquake magnitude $M = \log_{10}(A/A_0)$, exponential growth models



- To solve exponential equations for an unknown exponent, take the logarithm of both sides, then apply the Power Law

Exam Tips

- When converting to scientific notation, count decimal-point moves carefully -- moving LEFT (number was ≥ 10) gives a POSITIVE exponent, moving RIGHT (number was < 1) gives a NEGATIVE exponent
- Always find the characteristic FIRST using the digit-counting rule, then look up the mantissa separately in the table -- mixing up which part comes from which method is a common error
- Remember the mantissa is always written as positive, even when the characteristic is negative (shown with a bar, e.g. $\bar{2}.5433$) -- never treat the whole logarithm value as one single negative decimal
- When applying the laws of logarithm, work one law at a time: fully expand or combine using one law before applying the next, rather than trying to do multiple steps in your head at once
- For change-of-base problems, pick the new base a to be whatever makes the calculation easiest -- usually 10 (common log) or e (natural log), since those are the ones with tables/calculators
- For real-life logarithm problems (decibels, earthquakes, growth models), identify which given values correspond to which symbol in the formula before substituting -- writing them out explicitly avoids sign and placement errors



Unit 3: Sets and Functions

A set is a well-defined collection of distinct objects, described in descriptive, tabular (roster), or set-builder form. Building on this foundation, this unit reviews the key set types (singleton, empty, equal, equivalent, subset, universal, power set) and the operations performed on sets -- union, intersection, difference, and complement -- all visualized using Venn diagrams, developed by John Venn. It extends these ideas to three sets, covering the associative, distributive, and De Morgan's laws, and applies the inclusion-exclusion principle to solve real-world classification problems in demographics, surveys, and product categorization.

The unit then introduces binary relations (subsets of a Cartesian product $A \times B$) and functions -- a special type of relation where every element of the domain maps to exactly one element of the range. Relations and functions can be represented as tables, ordered pairs, or graphs. The unit classifies functions into into, one-one (injective), onto (surjective), and bijective types using mapping diagrams, and covers function notation, evaluation, and linear/quadratic function forms.

Learning Objectives

- Identify sets and apply union, intersection, and complement operations on two and three sets using Venn diagrams
- Verify the commutative, associative, distributive, and De Morgan's laws for union and intersection
- Apply the inclusion-exclusion principle to solve real-world classification and survey problems
- Explain the Cartesian product, binary relations, and their domain and range
- Represent a relation as a table, a set of ordered pairs, and a graph
- Determine the value, domain, and range of a function using function notation
- Identify into, one-one (injective), onto (surjective), and bijective functions using Venn/mapping diagrams



Key Concepts

3.1 Sets: Definitions and Representation

A set is a well-defined collection of distinct objects; capital letters name sets, lowercase letters name elements, and $\in$ denotes membership. A set can be described descriptively (in words), in tabular/roster form (listing elements), or by set-builder notation (e.g. $A = \{x \mid x \text{ is a vowel of the English alphabet}\}$).

Key set types: a singleton set has exactly one element; an empty (null/void) set $\emptyset$ has no elements; equal sets have exactly the same elements; equivalent sets have the same number of elements (written $A \sim B$); A is a subset of B ($A \subseteq B$) if every element of A is in B, a proper subset ($A \subset B$) if B also has an element not in A; the universal set U contains all elements under consideration; the power set $P(S)$ contains all subsets of S, with $n(P(S)) = 2^m$ for a set of m elements.

3.2 Operations on Sets and Venn Diagrams

Union $A \cup B = \{x \mid x \in A \vee x \in B\}$; Intersection $A \cap B = \{x \mid x \in A \wedge x \in B\}$; sets are disjoint if $A \cap B = \emptyset$, and overlapping if $A \cap B \neq \emptyset$ but neither is a subset of the other. Difference $A - B = \{x \mid x \in A \wedge x \notin B\}$; Complement $A' = U - A = \{x \mid x \in U \wedge x \notin A\}$. Venn diagrams (rectangle=U, circles=sets) visualize all these operations.

For three sets, the key properties are: commutative ($A \cup B = B \cup A$, $A \cap B = B \cap A$), associative ($A \cup (B \cap C) = (A \cup B) \cap C$), distributive ($A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and its dual), and De Morgan's Laws ($(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$) -- each verifiable both analytically (by listing elements) and visually (by shading Venn diagram regions).

3.2.1 The Inclusion-Exclusion Principle

The cardinality $n(A)$ of a set is its number of elements. For two sets, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$; for three sets, $n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$. These formulas solve real-world classification problems: surveys, demographic overlaps, and product categorization, letting unknown overlap counts be found from known totals.



3.3 Binary Relations

For non-empty sets A and B , the Cartesian product $A \times B = \{(x, y) \mid x \in A, y \in B\}$ is the set of all ordered pairs. Any subset of $A \times B$ is called a binary relation (or simply relation) from A to B , usually denoted r . The domain of r is the set of first elements of its ordered pairs; the range is the set of second elements. A relation can be represented as a table, a set of ordered pairs, or a graph (plotting each ordered pair as a point).

3.3.2 Functions: Definition, Notation, and Domain/Range

A function f from A to B is a relation (subset of $A \times B$) where the domain of f equals A , and no two ordered pairs share the same first element -- written $f: A \rightarrow B$. If $(x, y) \in f$, we write $y = f(x)$, calling y the value/image of x under f . The domain is the input set, the codomain is B , and the range ($\subseteq$ codomain) is the set of actual output values.

Function evaluation substitutes a specific input into the function's rule to compute the output, e.g. for $f(x) = 2x - 1$, $f(3) = 2(3) - 1 = 5$. A linear function has the form $\{(x, y) \mid y = mx + c\}$ (graph is a straight line); a quadratic function has the form $\{(x, y) \mid y = ax^2 + bx + c\}$.

3.3.2 Types of Functions

An into function $f: A \rightarrow B$ has $\text{Range}(f) \subset B$ (not every element of B is used). A one-one (injective) function has no two ordered pairs sharing the same second element. An onto (surjective) function has $\text{Range}(f) = B$ (every element of B is used). A bijective function is both one-one and onto, giving a (1-1) correspondence between A and B -- these types are identified visually using mapping/Venn diagrams.

Important Definitions

What is a set?

A well-defined collection of distinct objects, numbers, or elements.

What is a singleton set?

A set with exactly one element, e.g. $\{3\}$.



What is a subset?

$A \subseteq B$ if every element of A is also an element of B.

What is the power set of a set S?

$P(S)$, the set of all possible subsets of S; if $n(S)=m$, then $n(P(S))=2^m$.

What is the union of two sets A and B?

$A \cup B = \{x \mid x \in A \vee x \in B\}$, the set of all elements in A or B (or both).

What is the intersection of two sets A and B?

$A \cap B = \{x \mid x \in A \wedge x \in B\}$, the set of all elements in both A and B.

What is the Cartesian product $A \times B$?

The set of all ordered pairs (x,y) such that $x \in A$ and $y \in B$.

What is a binary relation from A to B?

Any subset of the Cartesian product $A \times B$.

What is a function $f:A \rightarrow B$?

A relation from A to B where the domain equals A and no two ordered pairs share the same first element.

What is a bijective function?

A function that is both one-one (injective) and onto (surjective), giving a (1-1) correspondence between the two sets.

Key Facts and Relations

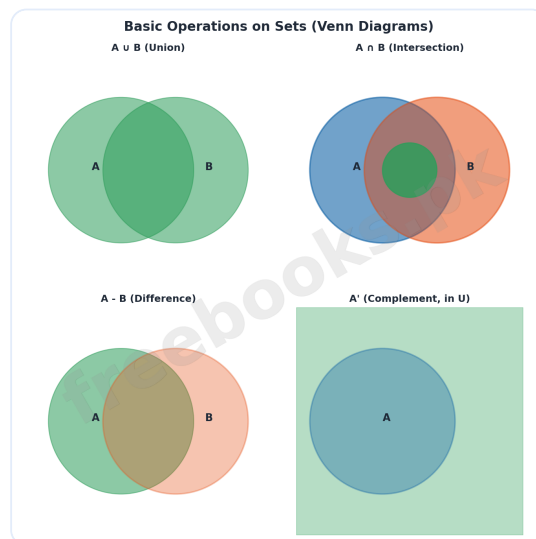
Topic	Key Fact / Relation
Union of two sets	$A \cup B = \{x \mid x \in A \vee x \in B\}$
Intersection of two sets	$A \cap B = \{x \mid x \in A \wedge x \in B\}$
Complement of a set	$A' = U - A = \{x \mid x \in U \wedge x \notin A\}$
De Morgan's Laws	$(A \cup B)' = A' \cap B'$; $(A \cap B)' = A' \cup B'$
Inclusion-exclusion, two sets	$n(A \cup B) = n(A) + n(B) - n(A \cap B)$
Inclusion-exclusion, three sets	$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C) + n(A \cap B \cap C)$



Topic	Key Fact / Relation
Cartesian product	$A \times B = \{(x,y) \mid x \in A, y \in B\}$
Power set size	$n(P(S)) = 2^m$, where $n(S)=m$
Linear function	$\{(x,y) \mid y = mx + c\}$
Quadratic function	$\{(x,y) \mid y = ax^2 + bx + c\}$

Diagrams

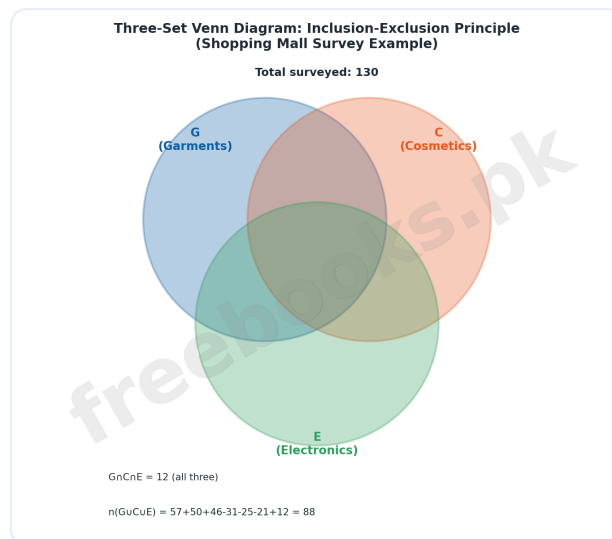
Basic Operations on Sets (Venn Diagrams): Four Venn diagrams showing union, intersection, difference, and complement of sets A and B relative to a universal set U



Basic Operations on Sets (Venn Diagrams)

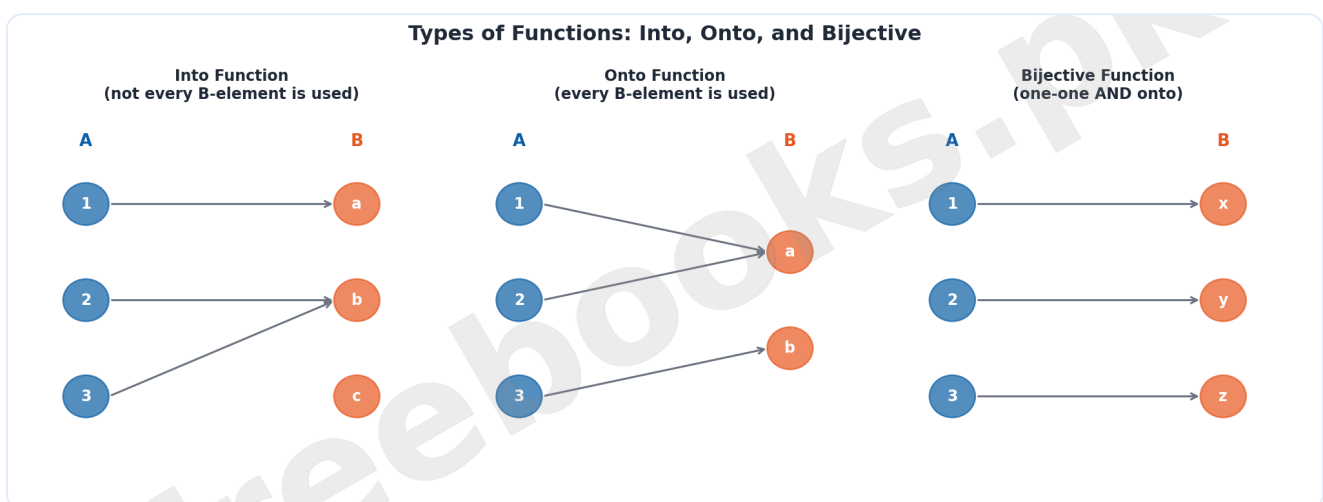
Three-Set Venn Diagram: Inclusion-Exclusion Principle: A three-circle Venn diagram illustrating the shopping-mall survey example, showing how the inclusion-exclusion formula combines individual, pairwise, and triple-overlap counts





Three-Set Venn Diagram: Inclusion-Exclusion Principle

Types of Functions: Into, Onto, and Bijective: Three mapping diagrams comparing an into function, an onto function, and a bijective function between sets A and B



Types of Functions: Into, Onto, and Bijective

Solved Examples

Example 1: Finding Union, Intersection, and Difference from a Venn Diagram

Problem: Given $A = \{1, 2, 3, 4, 5, 6, 8\}$ and $B = \{5, 6, 7, 8, 9, 10, 12\}$, find $A \cap B$, $B - A$, and $A \cup B$.

- Find $A \cap B$: the elements common to both sets are $\{5, 6, 8\}$.



2. Find $B-A$: the elements in B but not in A are $\{7,9,10,12\}$.
3. Find $A \cup B$: combine all elements from both sets without repetition.
4. List all elements from A and B together: $\{1,2,3,4,5,6,7,8,9,10,12\}$.
5. Final answer: $A \cap B = \{5,6,8\}$, $B-A = \{7,9,10,12\}$,
 $A \cup B = \{1,2,3,4,5,6,7,8,9,10,12\}$.

Example 2: Verifying De Morgan's Law

Problem: For $A = \{1,2,3\}$, $B = \{2,3,4,5\}$, and $U = \{1,2,\dots,10\}$, verify $(A \cup B)' = A' \cap B'$.

1. Compute $A \cup B$: $\{1,2,3\} \cup \{2,3,4,5\} = \{1,2,3,4,5\}$.
2. Compute $(A \cup B)'$: $U - (A \cup B) = \{6,7,8,9,10\}$.
3. Compute A' and B' separately: $A' = \{4,5,6,7,8,9,10\}$, $B' = \{1,6,7,8,9,10\}$.
4. Compute $A' \cap B'$: the common elements of A' and B' are $\{6,7,8,9,10\}$.
5. Final answer: $(A \cup B)' = \{6,7,8,9,10\} = A' \cap B' = \{6,7,8,9,10\}$, so De Morgan's Law is verified.

Example 3: Applying Inclusion-Exclusion for Two Sets

Problem: 98 students are in a sports club; 58 join swimming, 50 join tug-of-war. Find how many joined both.

1. Let A =swimming students, B =tug-of-war students; given $n(A \cup B) = 98$, $n(A) = 58$, $n(B) = 50$.
2. We need $n(A \cap B)$, the number who joined both clubs.
3. Rearrange the inclusion-exclusion formula: $n(A \cap B) = n(A) + n(B) - n(A \cup B)$.
4. Substitute: $n(A \cap B) = 58 + 50 - 98$.
5. Final answer: $n(A \cap B) = 10$ students joined both clubs.



Example 4: Applying Inclusion-Exclusion for Three Sets

Problem: 45 teachers teach math, physics, or chemistry: 18 math, 12 physics, 8 chemistry, 6 both math&physics, 4 both physics&chemistry, 2 both math&chemistry. Find how many teach all three.

1. Let M, P, C be the sets of teachers for each subject; given $n(M \cup P \cup C) = 45$, $n(M) = 18$, $n(P) = 12$, $n(C) = 8$, $n(M \cap P) = 6$, $n(P \cap C) = 4$, $n(M \cap C) = 2$.
2. We need $n(M \cap P \cap C)$, using the three-set inclusion-exclusion formula.
3. Rearrange: $n(M \cap P \cap C) = n(M \cup P \cup C) - n(M) - n(P) - n(C) + n(M \cap P) + n(P \cap C) + n(M \cap C)$.
4. Substitute: $n(M \cap P \cap C) = 45 - 18 - 12 - 8 + 6 + 4 + 2$.
5. Final answer: $n(M \cap P \cap C) = 19$ teachers teach all three subjects.

Example 5: Finding a Relation's Domain and Range

Problem: Let $A = \{1, 2, 3\}$. Determine the relation r such that $x r y$ iff $x < y$, and find its domain and range.

1. List all ordered pairs in $A \times A$: $(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3)$.
2. Apply the condition $x < y$ to select only pairs where the first element is less than the second.
3. The qualifying pairs are: $(1, 2), (1, 3), (2, 3)$.
4. Extract the domain (first elements) and range (second elements) from these pairs.
5. Final answer: $r = \{(1, 2), (1, 3), (2, 3)\}$, $\text{Dom}(r) = \{1, 2\}$, $\text{Range}(r) = \{2, 3\}$.



Example 6: Evaluating a Function and Identifying Its Domain, Codomain, and Range

Problem: If $A=\{0,1,2,3,4\}$, $B=\{3,5,7,9,11\}$, and $f:A\rightarrow B$ defined by $f=\{(x,y) \mid y=2x+3\}$, find f , $\text{Dom}(f)$, codomain, and $\text{Range}(f)$.

1. Substitute each value of x in A into $y=2x+3$ to generate ordered pairs.
2. For $x=0,1,2,3,4$: $y=3,5,7,9,11$ respectively.
3. Collect the ordered pairs: $f=\{(0,3),(1,5),(2,7),(3,9),(4,11)\}$.
4. Identify $\text{Dom}(f)$ as the set of first elements (equal to A), and the codomain as B .
5. Final answer: $\text{Dom}(f)=\{0,1,2,3,4\}=A$, $\text{Codomain}(f)=B$, $\text{Range}(f)=\{3,5,7,9,11\}\subseteq B$.

Example 7: Solving for Unknown Constants in a Linear Function

Problem: Given $f(x)=ax+b+3$, if $f(1)=4$ and $f(5)=9$, find a and b .

1. Substitute $x=1$ into $f(x)=ax+b+3=4$: $a+b+3=4$, giving $a+b=1$...(i).
2. Substitute $x=5$ into $f(x)=ax+b+3=9$: $5a+b+3=9$, giving $5a+b=6$...(ii).
3. Subtract equation (i) from (ii): $(5a+b)-(a+b)=6-1$, giving $4a=5$, so $a=5/4$.
4. Substitute $a=5/4$ back into equation (i): $5/4+b=1$, so $b=1-5/4=-1/4$.
5. Final answer: $a=5/4$ and $b=-1/4$.

Example 8: Real-Life Application: Three-Set Survey Analysis

Problem: 130 shopping mall customers: 57 garments, 50 cosmetics, 46 electronics, 31 garments&cosmetics, 25 garments&electronics, 21 cosmetics&electronics, 12 all three. Find how many bought at least one product.

1. Let G , C , E represent garments, cosmetics, and electronics buyers respectively.



2. Apply the three-set inclusion-exclusion formula: $n(G \cup C \cup E) = n(G) + n(C) + n(E) - n(G \cap C) - n(G \cap E) - n(C \cap E) + n(G \cap C \cap E)$.
3. Substitute the given values: $n(G \cup C \cup E) = 57 + 50 + 46 - 31 - 25 - 21 + 12$.
4. Compute step by step: $57 + 50 + 46 = 153$; $153 - 31 - 25 - 21 = 76$; $76 + 12 = 88$.
5. Final answer: 88 customers bought at least one of the three products.

Short Questions & Answers

What are the three ways to describe a set?

Descriptive form (in words), tabular/roster form (listing elements), and set-builder notation.

What is the difference between a subset and a proper subset?

A subset ($A \subseteq B$) allows $A = B$; a proper subset ($A \subset B$) requires B to have at least one element not in A .

When are two sets called disjoint?

When their intersection is the empty set, $A \cap B = \emptyset$.

What is the cardinality of a set?

The total number of elements in the set, denoted $n(A)$ for a set A .

How is the domain of a relation defined?

The set of first elements of all the ordered pairs forming the relation.

What distinguishes a function from a general relation?

In a function, the domain equals the starting set A , and no two ordered pairs share the same first element (each input maps to exactly one output).

What condition makes a function onto (surjective)?

$\text{Range}(f) = B$, meaning every element of the codomain B is the image of some element of A .



Long Questions & Answers

Explain the fundamental operations on sets (union, intersection, difference, complement) and how the properties of union and intersection are verified for three sets.

How are union, intersection, difference, and complement defined for two sets?

Union $A \cup B$ contains elements in A or B (or both); intersection $A \cap B$ contains elements in both A and B; difference $A - B$ contains elements in A but not B; complement A' (relative to universal set U) contains elements of U not in A -- each captures a different way of combining or contrasting two sets.

What are the commutative and associative properties of union and intersection?

Commutative: $A \cup B = B \cup A$ and $A \cap B = B \cap A$ (order doesn't matter); Associative: $A \cup (B \cap C) = (A \cup B) \cap C$ and $A \cap (B \cup C) = (A \cap B) \cup C$ (grouping doesn't matter) -- both can be verified either by listing out elements of each side or by shading the corresponding Venn diagram regions and comparing them.

What do the distributive properties state, and how are they verified?

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and its dual $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$; verification proceeds by computing both sides independently (either analytically with actual set elements, or visually by shading the described region in a Venn diagram) and confirming the resulting sets match exactly.

What do De Morgan's Laws state, and why are they useful?

$(A \cup B)' = A' \cap B'$ and $(A \cap B)' = A' \cup B'$ -- they let the complement of a combined set be rewritten in terms of the complements of the individual sets, which is essential for simplifying set expressions and solving classification problems where working with complements directly would be more complex.



Explain what a function is, how its domain, codomain, and range are identified, and how the four main types of functions (into, one-one, onto, bijective) are distinguished.

What conditions must a relation satisfy to be called a function?

A relation f from A to B is a function if its domain equals A (every element of A appears as a first element) and no two ordered pairs share the same first element -- meaning each input in A maps to exactly one output, written $f:A \rightarrow B$.

How are the domain, codomain, and range of a function identified?

The domain is the set of all first elements (equal to A by definition); the codomain is the target set B as stated in $f:A \rightarrow B$; the range is the actual set of second elements produced, which may be equal to or a proper subset of the codomain.

What distinguishes an into function from a one-one (injective) function?

An into function has $\text{Range}(f) \subset B$, meaning not every element of B is used as an output; a one-one (injective) function instead requires that no two ordered pairs share the same second element, meaning distinct inputs always produce distinct outputs -- these are independent properties describing different aspects of the mapping.

What distinguishes an onto (surjective) function from a bijective function?

An onto (surjective) function has $\text{Range}(f) = B$, meaning every element of B is used as an output by at least one input; a bijective function satisfies both the one-one and onto conditions simultaneously, giving a perfect (1-1) correspondence where every element of A maps to a unique element of B and every element of B is used exactly once.

Multiple Choice Questions (MCQs)

If $A = \{ \}$, then $P(A)$ is:

- A. $\{ \}$
- B. $\{1\}$



C. $\{\{\}\}$

D. $\emptyset$

Answer: C. $\{\{\}\}$

Explanation: The power set of the empty set contains exactly one element: the empty set itself, so $P(\emptyset) = \{\emptyset\}$, written $\{\{\}\}$.

If $U = \{1,2,3,4,5\}$, $A = \{1,2,3\}$, $B = \{3,4,5\}$, then $U - (A \cap B)$ is:

A. $\{1,2,4,5\}$

B. $\{2,3\}$

C. $\{1,3,4,5\}$

D. $\{1,2,3\}$

Answer: A. $\{1,2,4,5\}$

Explanation: $A \cap B = \{3\}$, so $U - (A \cap B) = U - \{3\} = \{1,2,4,5\}$.

If A and B are overlapping sets, then $n(A \cap B)$ is:

A. Always 0

B. Always equal to $n(A)$

C. Always positive but not equal to $n(A)$ or $n(B)$

D. Undefined

Answer: C. Always positive but not equal to $n(A)$ or $n(B)$

Explanation: Overlapping sets have a non-empty intersection, but neither is a subset of the other, so $n(A \cap B)$ is positive but strictly less than both $n(A)$ and $n(B)$.

If $n(A \cup B) = 50$, $n(A) = 30$, $n(B) = 35$, then $n(A \cap B)$ equals:

A. 23

B. 15

C. 9

D. 40

Answer: B. 15

Explanation: $n(A \cap B) = n(A) + n(B) - n(A \cup B) = 30 + 35 - 50 = 15$.

If $A = \{1,2,3,4\}$ and $B = \{x,y,z\}$, then $A \times B$ contains exactly:

A. 13 elements

B. 12 elements

C. 10 elements

D. 6 elements

Answer: B. 12 elements

Explanation: The Cartesian product has $n(A) \times n(B) = 4 \times 3 = 12$ ordered pairs.



Given $f(x)=3x+1$, if $f(x)=28$, then x equals:

- A. 9
- B. 27
- C. 3
- D. 18

Answer: A. 9

Explanation: Solving $3x+1=28$ gives $3x=27$, so $x=9$.

Let $A=\{1,2,3\}$, $B=\{a,b\}$ and $f=\{(1,a),(2,b),(3,b)\}$. This function is:

- A. Injective
- B. Surjective (onto)
- C. Bijective
- D. Into only

Answer: B. Surjective (onto)

Explanation: Since $\text{Range}(f)=\{a,b\}=B$, every element of B is used, making f onto (surjective); it is not injective since 2 and 3 both map to b .

The set of all subsets of a set S is called its:

- A. Universal set
- B. Complement
- C. Power set
- D. Cartesian product

Answer: C. Power set

Explanation: The power set $P(S)$ contains every possible subset of S , including $\emptyset$ and S itself.

A relation $f:A \rightarrow B$ is a function only if:

- A. $\text{Range}(f)=B$
- B. $\text{Domain}(f)=A$ and no two pairs share a first element
- C. A and B have the same number of elements
- D. f is a subset of $B \times A$

Answer: B. $\text{Domain}(f)=A$ and no two pairs share a first element

Explanation: The defining conditions of a function are that its domain equals A and each input maps to exactly one output.

A function that is both one-one and onto is called:

- A. Into
- B. Injective only



C. Surjective only

D. Bijective

Answer: D. Bijective

Explanation: A bijective function satisfies both the injective (one-one) and surjective (onto) conditions simultaneously.

Quick Revision Summary

- A set is a well-defined collection of distinct objects; described descriptively, in tabular form, or by set-builder notation
- Subset $A \subseteq B$: every element of A is in B; Power set $P(S)$ has 2^m elements for $n(S)=m$
- Union $A \cup B$, Intersection $A \cap B$, Difference $A - B$, Complement $A' = U - A$ -- all visualized with Venn diagrams
- Properties: commutative, associative, distributive laws, and De Morgan's Laws $(A \cup B)' = A' \cap B'$, $(A \cap B)' = A' \cup B'$
- Inclusion-exclusion (2 sets): $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ | (3 sets): add all three, subtract pairwise, add back the triple overlap
- Cartesian product $A \times B$ = all ordered pairs (x, y) , $x \in A$, $y \in B$; a binary relation is any subset of $A \times B$
- Domain = first elements of a relation's ordered pairs; Range = second elements
- A function $f: A \rightarrow B$ has domain $= A$ and no two pairs sharing a first element; $y = f(x)$ is the image of x
- Types of functions: into ($\text{Range} \subset B$), one-one/injective (distinct inputs $\rightarrow$ distinct outputs), onto/surjective ($\text{Range} = B$), bijective (both one-one and onto)
- Linear function: $y = mx + c$ (straight line) | Quadratic function: $y = ax^2 + bx + c$

Exam Tips

- When verifying set properties (commutative, associative, distributive, De Morgan's), always compute BOTH sides completely and independently before comparing -- don't assume equality partway through
- For inclusion-exclusion word problems, carefully identify which given number is the TOTAL (union), which are individual set sizes, and which are pairwise/triple overlaps before substituting into the formula



- Remember $A-B$ and $B-A$ are generally different sets (unless A and B are disjoint or one contains all of the other's non-shared elements) -- never assume set difference is symmetric
- When identifying a relation's domain and range, always double-check whether you're listing UNIQUE first/second elements, not just copying the ordered pairs directly
- To classify a function's type, check one-one and onto SEPARATELY -- a function can be one without being the other, and only functions satisfying both are bijective
- When solving for unknown constants in a linear function like $f(x)=ax+b+c$, substitute each given condition to form a system of two equations, then solve by elimination or substitution just as with any two-variable linear system



Unit 4: Factorization and Algebraic Manipulation

Factorization is the process of breaking a complex algebraic expression into a product of simpler expressions, and it is one of the most powerful tools in algebra. Once an expression is written as a product of factors, it becomes far easier to simplify, solve equations, find highest common factors and least common multiples, and even locate the roots or turning points of a real-world model. This unit builds the skill from the ground up — starting with concrete algebra tiles that let common factors and trinomials be seen physically, then moving to symbolic techniques for quadratics, higher-degree expressions, and special product forms.

Beyond pure manipulation, factorization has genuine practical value. Engineers use factorized cost and deflection functions to locate break-even points and points of zero stress; businesses use factorized profit functions to find the production level that maximizes return; and physicists use factorized potential-energy expressions to identify equilibrium points. This unit closes with several such applied examples, showing that factorization is not just an abstract classroom exercise but a working tool for real decisions.

Learning Objectives

- Identify common factors and factor trinomials concretely (using algebra tiles), pictorially, and symbolically.
- Factorize quadratic expressions of the form $x^2 + px + q$ and $ax^2 + bx + c$.
- Factorize expressions of the type $a^4 + a^2b^2 + b^4$ and $a^4 + b^4$.
- Factorize compound expressions such as $(ax^2 + bx + c)(ax^2 + bx + d) + k$ and $(x + a)(x + b)(x + c)(x + d) + k$ using substitution.
- Factorize perfect cubes $a^3 \pm 3a^2b + 3ab^2 \pm b^3$ and sums/differences of cubes $a^3 \pm b^3$.
- Find the HCF and LCM of algebraic expressions by factorization and by division, and use the relationship $\text{LCM} \times \text{HCF} = \text{product of the two expressions}$.



- Find the square root of an algebraic expression by factorization and by division method.
- Apply factorization of quadratic and cubic expressions to real-world problems in engineering, physics, and finance.

Key Concepts

4.1 Common Factors and Trinomial Factoring

A common factor is an expression that divides two or more terms exactly — for example, in $2x - 6 = 2(x - 3)$, the number 2 is the common factor. To build intuition before jumping into symbols, algebra tiles represent terms physically: a unit tile represents 1 (or -1), a rectangular tile represents x (or $-x$), and a squared tile represents x^2 (or $-x^2$). Arranging tiles into a rectangle whose side lengths are the two binomial factors makes trinomial factoring visible rather than purely mechanical.

A trinomial has three terms (like $x^2 - 5x + 4$) and factors into a product of two binomials (like $(x - 1)(x - 4)$). When a rectangle cannot be completed with the tiles given, a zero pair — equal numbers of opposite tiles that cancel to zero — can be added without changing the value of the expression, which then allows the rectangle to be completed.

4.1.3 Type I: $x^2 + px + q$ and $ax^2 + bx + c$

For $x^2 + px + q$, find two numbers whose product is q and whose sum is p , then split the middle term and factor by grouping. For example, $x^2 + 9x + 14$ splits as $x^2 + 2x + 7x + 14 = x(x + 2) + 7(x + 2) = (x + 2)(x + 7)$, since $2 \times 7 = 14$ and $2 + 7 = 9$.

When the coefficient of x^2 is not 1, as in $ax^2 + bx + c$, first multiply a and c , list factor pairs of that product, find the pair whose sum equals b , split the middle term using that pair, and factor by grouping. For $2x^2 + 17x + 26$: $a \cdot c = 52$, and the pair 4, 13 sums to 17, giving $2x^2 + 4x + 13x + 26 = 2x(x + 2) + 13(x + 2) = (x + 2)(2x + 13)$.



4.1.3 Type II-V: Higher-Degree and Special Forms

Type II expressions like $a^4 + a^2b^2 + b^4$ are factorized using the 'add and subtract' trick: rewrite as $(a^2 + b^2)^2 - a^2b^2 = (a^2 + b^2)^2 - (ab)^2$, which is a difference of squares, giving $(a^2 - ab + b^2)(a^2 + ab + b^2)$. The same idea handles $a^4 + b^4$ using $\sqrt{2}ab$ as the adjustment term.

Type III expressions like $(x + a)(x + b)(x + c)(x + d) + k$ are solved by regrouping factors so that the sums of the paired constants match (e.g., pairing so $a + d = b + c$), expanding each pair, substituting y for the repeated quadratic part, factoring the resulting quadratic in y , and substituting back.

Type IV expressions are perfect cubes: $a^3 + 3a^2b + 3ab^2 + b^3 = (a + b)^3$ and $a^3 - 3a^2b + 3ab^2 - b^3 = (a - b)^3$. Type V expressions are sum/difference of cubes: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ and $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

4.3 Highest Common Factor and Least Common Multiple

The HCF of two or more algebraic expressions is the greatest expression that divides each of them exactly, found either by factoring each expression and taking the product of common factors, or by repeated polynomial division. The LCM is the smallest expression divisible by each of the given expressions, found as the product of common factors and all non-common factors.

LCM and HCF are linked by the identity $\text{LCM} \times \text{HCF} = p(x) \times q(x)$, where $p(x)$ and $q(x)$ are the two original expressions. This relationship makes it possible to find one polynomial when the LCM, HCF, and the other polynomial are known — a technique used repeatedly in the worked examples and exercises.

4.4 Square Root of an Algebraic Expression

The square root of an algebraic expression is found either by factorization — rewriting the expression as a perfect square of a simpler expression and taking the square root of both sides — or by a long-division-style method that works term by term, which is especially useful when the expression's degree is too high to factor by inspection.



For example, $36x^4 - 36x^2 + 9 = 9(2x^2 - 1)^2 = [3(2x^2 - 1)]^2$, so its square root is $\pm 3(2x^2 - 1)$. The division method extends this same idea to expressions like $x^4 - 12x^3 + 42x^2 - 36x + 9$, whose square root is $\pm(x^2 - 6x + 3)$.

4.4.1 Real-World Applications

Factorization turns abstract cost, profit, and energy functions into forms that directly reveal meaningful points. A quadratic cost function $C(x) = 5x^2 - 25x + 30$ factorizes to $5(x - 2)(x - 3)$, showing the cost reaches its minimum between the roots $x = 2$ and $x = 3$. A cubic potential energy function $U(x) = x^3 - 6x^2 + 12x - 8$ factorizes to $(x - 2)^3$, revealing the single equilibrium point $x = 2$ directly.

For a profit function like $P(x) = -5x^2 + 50x - 120$, factorizing to $-5(x - 4)(x - 6)$ shows profit is zero at $x = 4$ and $x = 6$; because the parabola opens downward, maximum profit occurs at the midpoint $x = 5$. These examples show factorization is a practical shortcut for locating minimums, maximums, and break-even points without calculus.

Important Definitions

Common Factor

An expression that divides two or more given expressions exactly, without leaving a remainder.

Trinomial

An algebraic expression consisting of exactly three terms, such as $x^2 + 4x + 4$.

Binomial

An algebraic expression consisting of exactly two terms, such as $x + 2$.

Zero Pair

Two equal and opposite tiles (or terms) that sum to zero and can be added to or removed from an expression without changing its value.

Highest Common Factor (HCF)

The greatest algebraic expression that divides two or more given expressions exactly, without leaving a remainder.

Least Common Multiple (LCM)



The smallest algebraic expression that is exactly divisible by each of two or more given expressions.

Quadratic Expression

An algebraic expression of degree 2, such as $x^2 + px + q$.

Cubic Expression

An algebraic expression of degree 3, such as $a^3 + 3a^2b + 3ab^2 + b^3$.

Square Root of an Expression

An expression which, when multiplied by itself, produces the original given expression.

Sum/Difference of Cubes

The expressions $a^3 + b^3$ and $a^3 - b^3$, which factorize as $(a + b)(a^2 - ab + b^2)$ and $(a - b)(a^2 + ab + b^2)$ respectively.

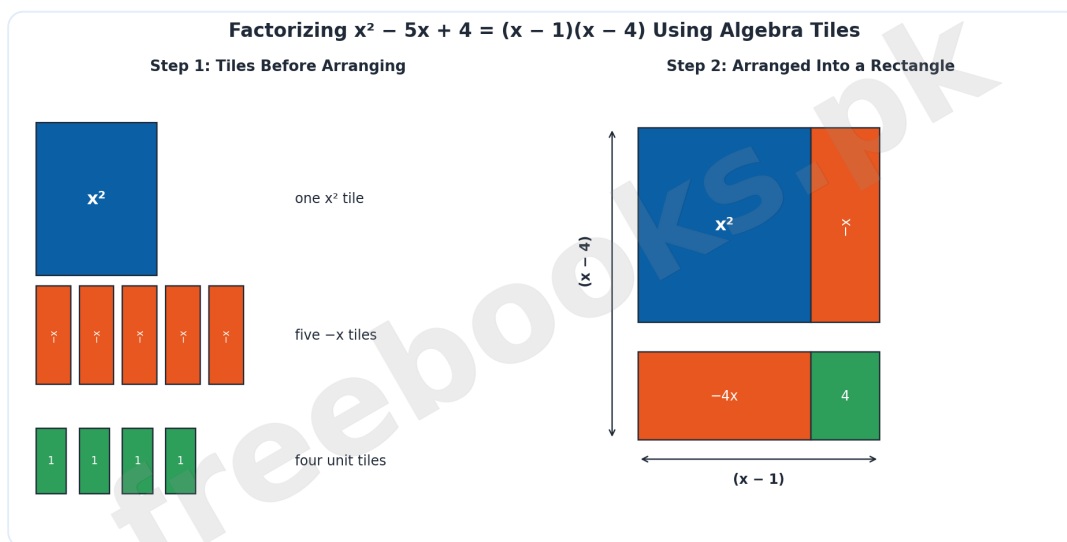
Key Facts and Relations

Topic	Key Fact / Relation
$a^2 - b^2 = (a - b)(a + b)$	Difference of squares identity.
$(a + b)^2 = a^2 + 2ab + b^2$	Square of a sum.
$(a - b)^2 = a^2 - 2ab + b^2$	Square of a difference.
$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$	Cube of a sum.
$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$	Cube of a difference.
$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$	Sum of cubes.
$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$	Difference of cubes.
$a^4 + a^2b^2 + b^4 = (a^2 - ab + b^2)(a^2 + ab + b^2)$	Add-and-subtract technique for this special quartic form.
$\text{LCM} \times \text{HCF} = p(x) \times q(x)$	Relationship between LCM, HCF, and the two original polynomials.
$\text{LCM} = \text{Common factors} \times \text{Non-common factors}$	Formula for computing LCM by the factorization method.



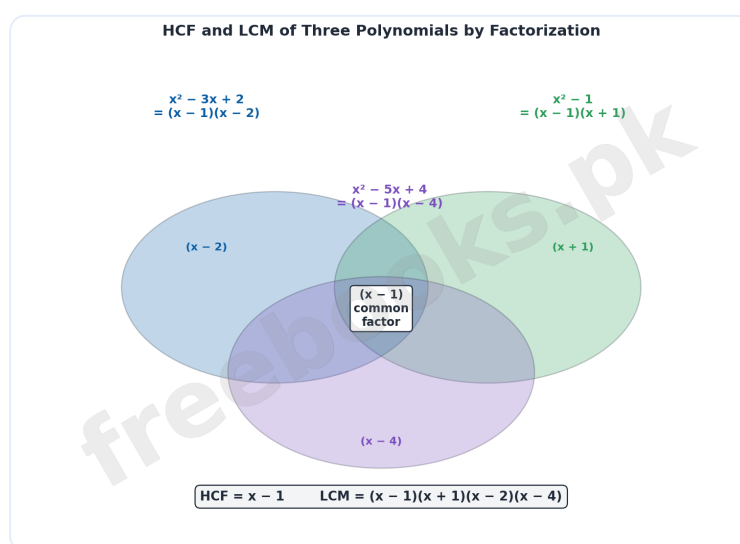
Diagrams

Concrete-to-Symbolic Factoring With Algebra Tiles: A tile diagram showing $x^2 - 5x + 4$ factorized as $(x - 1)(x - 4)$ by arranging unit, rectangular, and squared tiles into a rectangle



Concrete-to-Symbolic Factoring With Algebra Tiles

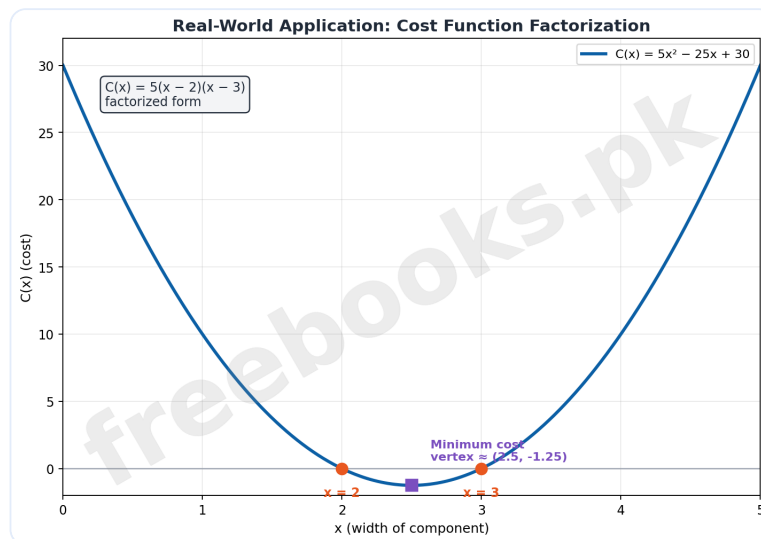
HCF and LCM by Factorization (Venn Diagram): A Venn-style diagram showing common and non-common factors of three polynomials, giving their HCF and LCM



HCF and LCM by Factorization (Venn Diagram)

Real-World Application: Cost Function Minimum: A graph of a factorized quadratic cost function showing its roots and minimum-cost vertex





Real-World Application: Cost Function Minimum

Solved Examples

Example 1: Factorizing $x^2 + 9x + 14$ by Splitting the Middle Term

Problem: Factorize $x^2 + 9x + 14$.

1. Find two numbers whose product is 14 and whose sum is 9: these are 2 and 7.
2. Split the middle term: $x^2 + 9x + 14 = x^2 + 2x + 7x + 14$.
3. Group and factor: $x(x + 2) + 7(x + 2)$.
4. Take the common binomial factor: $(x + 2)(x + 7)$.

Example 2: Factorizing $ax^2 + bx + c$ When $a \neq 1$

Problem: Factorize $2x^2 + 17x + 26$.

1. Multiply the coefficient of x^2 by the constant term: $2 \times 26 = 52$.
2. List factor pairs of 52 and find the pair summing to 17: 4 and 13.
3. Split the middle term: $2x^2 + 4x + 13x + 26$.
4. Group and factor: $2x(x + 2) + 13(x + 2) = (x + 2)(2x + 13)$.



Example 3: Factorizing $a^4 + 64$ Using the Add-and-Subtract Technique

Problem: Factorize $a^4 + 64$.

1. Write $a^4 + 64$ as $(a^2)^2 + (8)^2$.
2. Add and subtract $2(a^2)(8) = 16a^2$: $(a^2)^2 + (8)^2 + 16a^2 - 16a^2$.
3. Group as a perfect square minus a square: $(a^2 + 8)^2 - (4a)^2$.
4. Apply the difference of squares identity: $(a^2 + 8 - 4a)(a^2 + 8 + 4a) = (a^2 - 4a + 8)(a^2 + 4a + 8)$.

Example 4: Factorizing $(x+2)(x+3)(x+4)(x+5) - 15$ Using Substitution

Problem: Factorize $(x + 2)(x + 3)(x + 4)(x + 5) - 15$.

1. Rearrange the factors so paired constants have equal sums: $2 + 5 = 3 + 4$.
2. Pair and expand: $[(x+2)(x+5)][(x+3)(x+4)] - 15 = (x^2 + 7x + 10)(x^2 + 7x + 12) - 15$.
3. Let $y = x^2 + 7x$, giving $(y + 10)(y + 12) - 15 = y^2 + 22y + 105$.
4. Factorize the quadratic in y : $y^2 + 22y + 105 = (y + 15)(y + 7)$.
5. Substitute back $y = x^2 + 7x$: $(x^2 + 7x + 15)(x^2 + 7x + 7)$.

Example 5: Factorizing $8x^3 + 27$ as a Sum of Cubes

Problem: Factorize $8x^3 + 27$.

1. Write $8x^3 + 27$ as $(2x)^3 + (3)^3$.
2. Apply the sum-of-cubes identity $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$.
3. Substitute $a = 2x$, $b = 3$: $(2x + 3)[(2x)^2 - (2x)(3) + (3)^2]$.
4. Simplify: $(2x + 3)(4x^2 - 6x + 9)$.



Example 6: Finding HCF of Three Polynomials by Factorization

Problem: Find the HCF of $x^2 - 27$, $x^2 + 6x - 27$, and $x^2 - 9$.

1. Factorize the first expression: $x^3 - 27 = (x - 3)(x^2 + 3x + 9)$.
2. Factorize the second: $x^2 + 6x - 27 = x^2 + 9x - 3x - 27 = (x + 9)(x - 3)$.
3. Factorize the third: $x^2 - 9 = (x - 3)(x + 3)$.
4. Identify the factor common to all three expressions: $(x - 3)$.
5. State the result: $\text{HCF} = x - 3$.

Example 7: Finding LCM of Three Polynomials by Factorization

Problem: Find the LCM of $x^2 - 3x + 2$, $x^2 - 1$, and $x^2 - 5x + 4$.

1. Factorize each expression: $x^2 - 3x + 2 = (x - 2)(x - 1)$; $x^2 - 1 = (x - 1)(x + 1)$; $x^2 - 5x + 4 = (x - 4)(x - 1)$.
2. Identify the common factor across all three: $(x - 1)$.
3. Identify the remaining non-common factors: $(x + 1)$, $(x - 2)$, $(x - 4)$.
4. Multiply common and non-common factors together: $\text{LCM} = (x - 1)(x + 1)(x - 2)(x - 4)$.

Example 8: Real-World Application: Maximizing Profit by Factorization

Problem: A company's profit is $P(x) = -5x^2 + 50x - 120$. Factorize it and find the production level that maximizes profit.

1. Factor out -5 : $-5(x^2 - 10x + 24)$.
2. Split the middle term of the quadratic: $x^2 - 4x - 6x + 24 = x(x - 4) - 6(x - 4) = (x - 4)(x - 6)$.
3. Write the full factorization: $P(x) = -5(x - 4)(x - 6)$.
4. Profit is zero at $x = 4$ and $x = 6$; since the leading coefficient is negative, the maximum lies at the midpoint.



5. Compute the midpoint: $x = (4 + 6)/2 = 5$, so producing 5 units maximizes profit.

Short Questions & Answers

What is a common factor, and how is it identified in an expression like $2x - 6$?

A common factor is an expression that divides every term exactly. In $2x - 6 = 2(x - 3)$, the number 2 divides both $2x$ and 6 exactly, so it is the common factor.

What is the difference between a binomial and a trinomial?

A binomial has exactly two terms, such as $x + 2$, while a trinomial has exactly three terms, such as $x^2 + 4x + 4$.

Why is a 'zero pair' added when factoring some trinomials with algebra tiles?

A zero pair (equal numbers of opposite tiles) sums to zero, so adding one does not change the value of the expression — it is added only to allow the tiles to be arranged into a complete rectangle.

What is the first step in factoring $ax^2 + bx + c$ when a is not 1?

Multiply the coefficient a by the constant c , then find two factors of that product whose sum equals b , and use them to split the middle term.

State the identity used to factorize a sum of two cubes.

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$

How is the LCM of algebraic expressions calculated once each is factorized?

LCM = common factors $\times$ non-common factors, where common factors are taken once and non-common factors are all remaining factors from each expression.

What relationship links the LCM and HCF of two polynomials to the polynomials themselves?

$\text{LCM} \times \text{HCF} = p(x) \times q(x)$, the product of the two original polynomials.



Long Questions & Answers

Explain the different types of factorization covered in this unit, with an example of each.

What is Type I factorization?

Type I covers $x^2 + px + q$ and $ax^2 + bx + c$, factorized by splitting the middle term into two parts whose product matches ac and whose sum matches b , then grouping. For example, $x^2 + 9x + 14 = (x + 2)(x + 7)$.

What is Type II factorization?

Type II covers $a^4 + a^2b^2 + b^4$ and $a^4 + b^4$, factorized using the add-and-subtract technique to create a difference of squares. For example, $a^4 + 64 = (a^2 - 4a + 8)(a^2 + 4a + 8)$.

What is Type III factorization?

Type III covers compound products like $(x+a)(x+b)(x+c)(x+d) + k$, solved by regrouping factors with matching constant sums, substituting a variable for the repeated quadratic, and factoring the resulting quadratic before substituting back.

What are Types IV and V factorization?

Type IV covers perfect cubes such as $a^3 + 3a^2b + 3ab^2 + b^3 = (a + b)^3$. Type V covers sums and differences of cubes, $a^3 \pm b^3$, factorized using the standard cube identities into a linear factor times a quadratic factor.

Describe how HCF and LCM of algebraic expressions are found, and how they relate to each other.

How is HCF found by factorization?

Each expression is factorized fully into irreducible factors, and the HCF is the product of the factors common to all the given expressions.



How is HCF found by division method?

The larger-degree expression is divided by the smaller repeatedly (similar to Euclid's algorithm for numbers), and the last nonzero remainder, after removing any purely numerical common factor, gives the HCF.

How is LCM found once expressions are factorized?

LCM = common factors $\times$ non-common factors — the common factors are included once, and every factor unique to any of the expressions is also included.

What identity links LCM, HCF, and the original expressions?

$\text{LCM} \times \text{HCF} = p(x) \times q(x)$, the product of the two polynomials. This lets one polynomial be found algebraically if the LCM, HCF, and the other polynomial are known.

Multiple Choice Questions (MCQs)

The factorization of $12x + 36$ is:

- A. $12(x + 3)$
- B. $12(3x)$
- C. $12(3x + 1)$
- D. $x(12 + 36x)$

Answer: A. $12(x + 3)$

Explanation: 12 is the common factor of both terms: $12x + 36 = 12(x + 3)$.

The factors of $4x^2 - 12x + 9$ are:

- A. $(2x + 3)^2$
- B. $(2x - 3)^2$
- C. $(2x - 3)(2x + 3)$
- D. $(2 + 3x)(2 - 3x)^2$

Answer: B. $(2x - 3)^2$

Explanation: $4x^2 - 12x + 9 = (2x)^2 - 2(2x)(3) + 3^2 = (2x - 3)^2$.

The HCF of a^3b^3 and ab^2 is:

- A. a^3b^3
- B. ab^2



C. a^4b^5

D. a^2b

Answer: B. ab^2

Explanation: The lower power of each common variable is taken: a^1 and b^2 , giving $HCF = ab^2$.

Product of LCM and HCF equals the _____ of two polynomials.

A. sum

B. difference

C. product

D. quotient

Answer: C. product

Explanation: $LCM \times HCF$ always equals the product of the two original polynomials.

The square root of $x^2 - 6x + 9$ is:

A. $\pm(x - 3)$

B. $\pm(x + 3)$

C. $x - 3$

D. $x + 3$

Answer: A. $\pm(x - 3)$

Explanation: $x^2 - 6x + 9 = (x - 3)^2$, so its square root is $\pm(x - 3)$.

The LCM of $(a - b)^2$ and $(a - b)^4$ is:

A. $(a - b)^2$

B. $(a - b)^3$

C. $(a - b)^4$

D. $(a - b)^6$

Answer: C. $(a - b)^4$

Explanation: The LCM takes the higher power present, which is $(a - b)^4$.

Factorization of $x^3 + 3x^2 + 3x + 1$ is:

A. $(x + 1)^3$

B. $(x - 1)^3$

C. $(x + 1)(x^2 + x + 1)$

D. $(x - 1)(x^2 - x + 1)$

Answer: A. $(x + 1)^3$



Explanation: This matches the perfect-cube identity $a^3 + 3a^2b + 3ab^2 + b^3$ with $a = x$, $b = 1$, giving $(x + 1)^3$.

A cubic polynomial has degree:

- A. 1
- B. 2
- C. 3
- D. 4

Answer: C. 3

Explanation: By definition, a cubic expression has degree 3.

One of the factors of $x^3 - 27$ is:

- A. $x - 3$
- B. $x + 3$
- C. $x^2 - 3x + 9$
- D. Both $x - 3$ and $x^2 + 3x + 9$

Answer: D. Both $x - 3$ and $x^2 + 3x + 9$

Explanation: $x^3 - 27 = (x - 3)(x^2 + 3x + 9)$, the difference-of-cubes identity, so both are factors.

The two numbers whose product is 14 and sum is 9, used to factor $x^2 + 9x + 14$, are:

- A. 1 and 14
- B. 2 and 7
- C. 7 and 2 only
- D. -2 and -7

Answer: B. 2 and 7

Explanation: $2 \times 7 = 14$ and $2 + 7 = 9$, matching both required conditions.

Quick Revision Summary

- A common factor divides every term of an expression exactly; factor it out first before attempting other techniques.
- For $x^2 + px + q$, find two numbers with product q and sum p to split the middle term.
- For $ax^2 + bx + c$, multiply a and c first, then find factors of that product summing to b .



- $a^4 + a^2b^2 + b^4$ and $a^4 + b^4$ are solved using the add-and-subtract trick to form a difference of squares.
- Perfect cubes $(a \pm b)^3$ expand to $a^3 \pm 3a^2b + 3ab^2 \pm b^3$; sum/difference of cubes $a^3 \pm b^3$ factor into a linear times a quadratic factor.
- $\text{LCM} \times \text{HCF}$ always equals the product of the two original polynomials — a useful check and shortcut.

Exam Tips

- Always check for a common factor first before applying any other factorization technique — it simplifies everything that follows.
- When splitting the middle term of $ax^2 + bx + c$, always multiply $a \times c$ first, not just look at c alone.
- Memorize the cube identities $(a \pm b)^3$ and $a^3 \pm b^3$ — they appear constantly in Type IV and V problems.
- For compound expressions like $(x+a)(x+b)(x+c)(x+d) + k$, look for pairs of constants with equal sums before expanding.
- Verify a square root answer by squaring it back — it should exactly reproduce the original expression.
- In real-world problems, factoring reveals roots and turning points directly, often faster than other algebraic methods.



Unit 5: Linear Equations and Inequalities

Linear equations and inequalities are among the most widely used tools in mathematics for modeling real-world relationships and making decisions under constraints. A linear equation in one variable has exactly one solution, found by isolating the variable through balanced operations on both sides. A linear inequality, by contrast, describes a whole range of possible values rather than a single number — its solution set can be represented as an interval on a real line, or, when two variables are involved, as a region of the coordinate plane.

This unit builds from solving simple one-variable equations and inequalities to graphing inequalities in two variables as half planes, and finally to combining several inequalities into a system whose solutions form a feasible region. Every point in that region is a candidate solution to some real-world constraint problem, and the unit closes with the central technique of linear programming: evaluating an objective function at the corner points of the feasible region to find its maximum or minimum value — the same basic idea used in business, engineering, and resource-allocation problems everywhere.

Learning Objectives

- Solve linear equations and inequalities with rational coefficients and represent the solution set on a real line.
- Understand the effect of adding, subtracting, multiplying, and dividing both sides of an inequality, including the sign reversal caused by a negative multiplier or divisor.
- Graph a linear inequality in two variables as a half plane, using a test point to determine the correct shaded region.
- Solve two linear inequalities with two unknowns simultaneously by finding the intersection of their half planes.
- Interpret and identify the feasible region bounded by a system of linear inequalities in two unknowns.
- Identify the corner points (vertices) of a feasible region.



- Find the maximum and minimum values of an objective function by evaluating it at the corner points of the feasible region.

Key Concepts

5.1 Linear Equations in One Variable

A linear equation in one variable has the general form $ax + b = 0$, where a and b are constants and $a \neq 0$, with the highest power of the variable equal to 1. Such an equation always has exactly one solution.

Solving proceeds in four steps: simplify both sides by combining like terms and distributing over parentheses, isolate the variable term by moving all variable terms to one side and constants to the other, solve for the variable by dividing by its coefficient, and finally check the solution by substituting it back into the original equation.

5.2 Linear Inequalities in One and Two Variables

Inequalities use the symbols $>$, $<$, $\geq$, and $\leq$. Adding or subtracting a constant, or multiplying/dividing by a positive constant, does not change the direction of an inequality — but multiplying or dividing by a negative constant reverses it. A linear inequality in two variables has the form $ax + by < c$ (or $>$, $\leq$, $\geq$), and its associated equation $ax + by = c$ is a line that divides the plane into two half planes.

A vertical line divides the plane into left and right half planes; a non-vertical line divides it into upper and lower half planes. To graph an inequality: draw the associated equation as a dashed line for strict inequalities ($<$ or $>$) or a solid line for non-strict inequalities ($\leq$ or $\geq$), then pick a test point not on the line (commonly the origin) to determine which half plane satisfies the inequality.

5.2.2 Systems of Two Linear Inequalities

The graph of a system of linear inequalities is the set of all ordered pairs (x, y) that satisfy every inequality in the system simultaneously. This is found by graphing each inequality's half plane on the same axes and taking their



intersection — the overlapping shaded region is the solution region for the whole system.

5.3 Feasible Region and Corner Points

In applied problems, each linear inequality represents a problem constraint, and the full system is the set of problem constraints. Variables are typically restricted to non-negative values ($x \geq 0$, $y \geq 0$), called non-negative constraints, since they usually represent real quantities like units produced or hours worked. The feasible region is the solution region restricted to the first quadrant, and every point inside it is a feasible solution.

A corner point (or vertex) of the feasible region is a point where two of its boundary lines intersect. Corner points are found either by reading intersection points off the graph or by solving the relevant pairs of boundary equations simultaneously.

5.3.2 Optimizing an Objective Function

A function to be maximized or minimized, subject to the constraints of a feasible region, is called an objective function. Because a feasible region generally contains infinitely many points, the key theorem used in linear programming is that the optimal (maximum or minimum) value of a linear objective function always occurs at one of the corner points of the feasible region — never at an interior point.

The procedure is: graph the constraint inequalities to find the feasible region, identify all corner points, then evaluate the objective function at each corner point and compare the results — the largest value gives the maximum, the smallest gives the minimum.

Important Definitions

Linear Equation in One Variable

An equation of the form $ax + b = 0$, where $a \neq 0$, whose highest variable power is 1 and which has exactly one solution.

Linear Inequality



A statement relating two expressions using $>$, $<$, $\geq$, or $\leq$ instead of equality, whose solution set is generally a range of values rather than a single value.

Half Plane

One of the two regions into which a line $ax + by = c$ divides the coordinate plane; each region is the graph of a strict or non-strict linear inequality.

Associated (Corresponding) Equation

The linear equation $ax + by = c$ obtained by replacing the inequality symbol of $ax + by < c$ (or $>$, $\leq$, $\geq$) with an equal sign.

Test Point

A point not on the boundary line, used to determine which half plane is the solution region of an inequality; the origin $(0,0)$ is commonly used when it is not on the line.

Problem Constraint

A linear inequality representing a real-world restriction in an applied problem; the full set of such inequalities is called the problem constraints.

Feasible Region

The solution region of a system of linear inequalities, restricted to the first quadrant, containing all feasible solutions of the system.

Corner Point (Vertex)

A point of the feasible region where two of its boundary lines intersect.

Objective Function

A function, typically linear, that is to be maximized or minimized subject to a set of constraints.

Optimal Solution

The feasible solution — always found at a corner point — that produces the maximum or minimum value of the objective function.

Key Facts and Relations

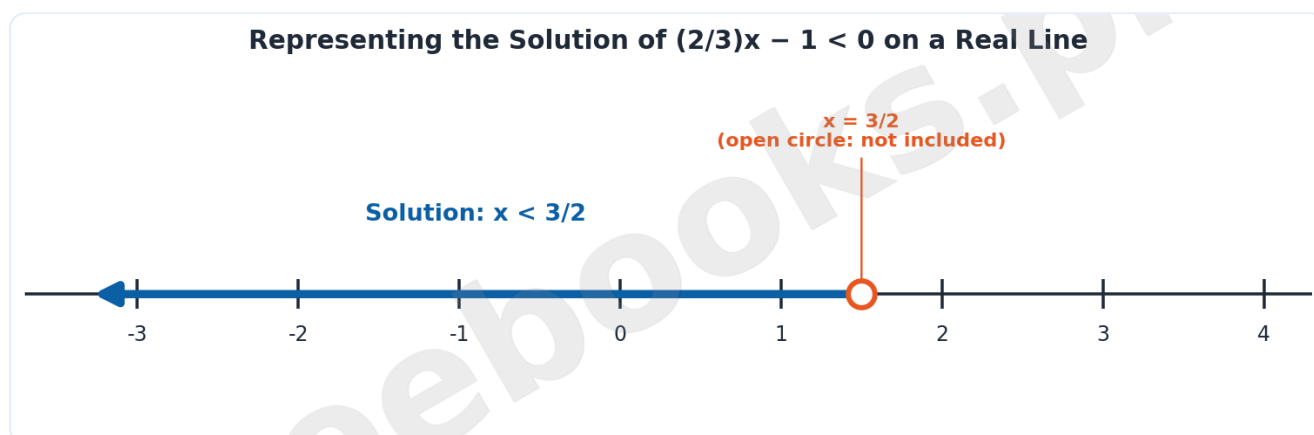
Topic	Key Fact / Relation
$ax + b = 0, a \neq 0$	General form of a linear equation in one variable.



Topic	Key Fact / Relation
$ax + by = c$	Associated equation of a linear inequality in two variables; a line dividing the plane into two half planes.
If $k < 0$: $a < b \Rightarrow ka > kb$	Multiplying or dividing an inequality by a negative constant reverses its direction.
Test point rule	Substitute the test point into the inequality: if true, its half plane is the solution; if false, the opposite half plane is the solution.
Feasible region = $\cap$ (half planes)	The feasible region is the intersection of all half planes defined by the system's constraints, restricted to $x \geq 0$, $y \geq 0$.
Optimal solution = max/min of f at corner points	The maximum or minimum of the objective function over the feasible region occurs at one of its corner points.

Diagrams

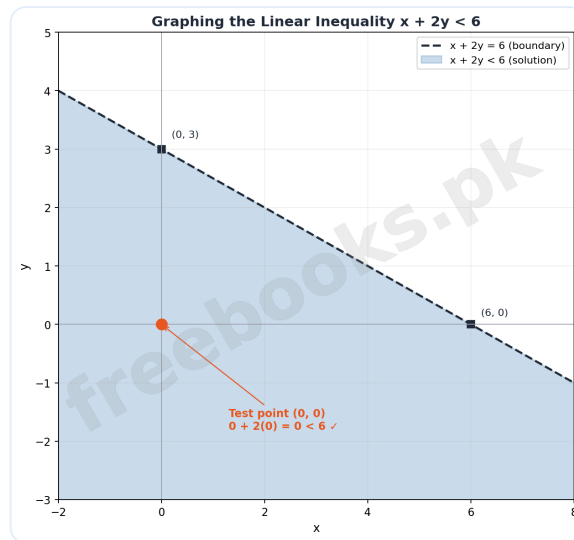
Representing an Inequality's Solution on a Real Line: A number-line diagram showing the open-circle boundary and shaded ray representing the solution $x < 3/2$ of a linear inequality



Representing an Inequality's Solution on a Real Line

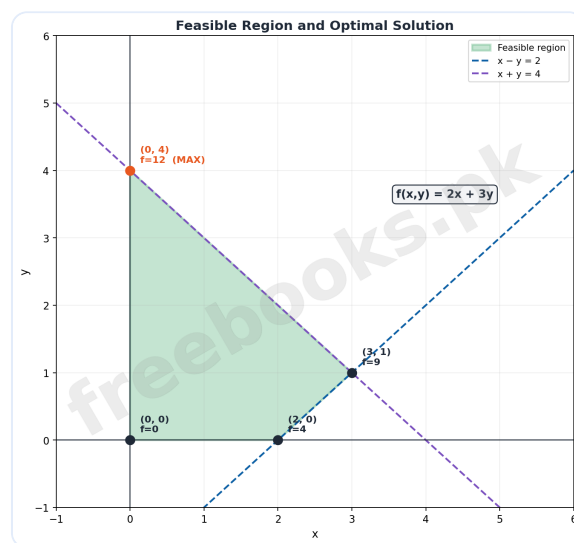
Graphing a Linear Inequality as a Half Plane: A shaded half-plane diagram for $x + 2y < 6$, showing the dashed boundary line, intercepts, and the origin used as a test point





Graphing a Linear Inequality as a Half Plane

Feasible Region and Optimal Solution: A shaded feasible region bounded by two constraint lines and the axes, with corner points labeled by their objective function values, showing the maximum at (0,4)



Feasible Region and Optimal Solution

Solved Examples

Example 1: Solving a Linear Equation With a Fractional Expression

Problem: Solve $(x - 2)/5 - (x - 4)/2 = 2$ and check the solution.

- Combine the fractions over a common denominator of 10: $[2(x-2) - 5(x-4)] / 10 = 2$.



2. Expand the numerator: $2x - 4 - 5x + 20 = -3x + 16$, so $(-3x + 16)/10 = 2$.
3. Multiply both sides by 10: $-3x + 16 = 20$.
4. Isolate x : $-3x = 4$, so $x = -4/3$.
5. Check by substituting $x = -4/3$ back into the original equation; both sides simplify to 2, confirming the solution.

Example 2: Solving and Graphing a One-Variable Inequality

Problem: Find the solution of $(2/3)x - 1 < 0$ and represent it on a real line.

1. Add 1 to both sides: $(2/3)x < 1$.
2. Multiply both sides by 3 (positive, so direction unchanged): $2x < 3$.
3. Divide both sides by 2: $x < 3/2$.
4. Represent the solution as the open interval $(-\infty, 3/2)$ on a real line, with an open circle at $3/2$ since it is not included.

Example 3: Graphing a Linear Inequality in Two Variables

Problem: Solve the inequality $x + 2y < 6$ by graphing its solution region.

1. Write the associated equation $x + 2y = 6$, which crosses the axes at $(6, 0)$ and $(0, 3)$.
2. Draw this line using dashes, since the inequality is strict ($<$).
3. Choose the origin $(0, 0)$ as a test point since it is not on the line.
4. Substitute: $0 + 2(0) = 0 < 6$, which is true, so the origin's side of the line is the solution region.
5. Shade the half plane containing the origin — the region below the line.



Example 4: Solving a System of Two Linear Inequalities

Problem: Find the solution region for the system $x - 2y \leq 6$ and $2x + y \geq 2$.

1. Graph the associated equation $x - 2y = 6$ using intercepts $(6, 0)$ and $(0, -3)$ as a solid line, since the inequality is non-strict.
2. Test the origin: $0 - 2(0) = 0 < 6$, true, so shade the half plane containing the origin.
3. Graph the associated equation $2x + y = 2$ using intercepts $(1, 0)$ and $(0, 2)$ as a solid line.
4. Test the origin: $2(0) + 0 = 0$, which is not ≥ 2 , so shade the half plane not containing the origin.
5. The solution region of the system is the overlap (intersection) of both shaded half planes.

Example 5: Finding the Feasible Region and Its Corner Points

Problem: Shade the feasible region for $x - y \leq 3$, $x + 2y \leq 6$, $x \geq 0$, $y \geq 0$, and find its corner points.

1. Graph $x - y = 3$ as a solid line through $(3, 0)$ and $(0, -3)$; test the origin ($0 < 3$, true) to shade the origin's side.
2. Graph $x + 2y = 6$ as a solid line through $(6, 0)$ and $(0, 3)$; test the origin ($0 < 6$, true) to shade the origin's side.
3. Restrict to the first quadrant by intersecting with $x \geq 0$ and $y \geq 0$.
4. The resulting feasible region is bounded by the four constraints, with corner points $(0, 0)$, $(3, 0)$, $(4, 1)$, and $(0, 3)$.



Example 6: Finding the Maximum and Minimum of an Objective Function

Problem: Find the maximum and minimum values of $f(x, y) = 2x + 3y$ subject to $x - y \leq 2$, $x + y \leq 4$, $x \geq 0$, $y \geq 0$.

1. Graph both constraint lines and restrict to the first quadrant to find the feasible region ABCD.
2. Identify the corner points: $(0, 0)$, $(2, 0)$, $(3, 1)$, and $(0, 4)$.
3. Evaluate f at each corner: $f(0,0) = 0$, $f(2,0) = 4$, $f(3,1) = 2(3)+3(1) = 9$, $f(0,4) = 2(0)+3(4) = 12$.
4. Compare the values: the minimum is 0 at $(0, 0)$ and the maximum is 12 at $(0, 4)$.

Example 7: Solving an Inequality Involving a Vertical Boundary Line

Problem: Solve the inequality $2x \geq -3$ in the xy -plane.

1. Rewrite as $2x + 0y \geq -3$ to treat it as a two-variable inequality.
2. Solve for x : $x \geq -3/2$, so the solution set is all points (x, y) with $x, y \in \mathbb{R}$ and $x \geq -3/2$.
3. The associated equation $2x = -3$ is a vertical line parallel to the y -axis.
4. The graph consists of the boundary line and the closed half plane to the right of it.

Example 8: Real-World Application: Maximizing Profit With Linear Programming

Problem: A furniture maker earns profit modeled by $f(x, y) = 3x + 5y$, where x and y are units of two products, subject to $2x + y \leq 10$, $x + 2y \leq 14$, $x \geq 0$, $y \geq 0$. Find the corner points and the production plan that maximizes profit.

1. Graph $2x + y = 10$ with intercepts $(5, 0)$ and $(0, 10)$; test the origin (true) to shade its side.



2. Graph $x + 2y = 14$ with intercepts $(14, 0)$ and $(0, 7)$; test the origin (true) to shade its side.
3. Restrict to the first quadrant and find the feasible region's corner points: $(0, 0)$, $(5, 0)$, $(2, 6)$, and $(0, 7)$.
4. Evaluate f at each: $f(0,0)=0$, $f(5,0)=15$, $f(2,6)=6+30=36$, $f(0,7)=35$.
5. The maximum profit of 36 occurs at $(2, 6)$, so the maker should produce 2 units of the first product and 6 of the second.

Short Questions & Answers

What is the general form of a linear equation in one variable?

$ax + b = 0$, where a and b are constants, $a \neq 0$, and the highest power of the variable x is 1.

How many solutions does a linear equation in one variable have?

Exactly one solution.

What operation reverses the direction of an inequality?

Multiplying or dividing both sides of the inequality by a negative constant.

What is a half plane?

One of the two regions into which a line divides the coordinate plane, representing the graph of a linear inequality in two variables.

Why is a dashed line used for a strict inequality like $x + 2y < 6$?

Because no point on the boundary line itself satisfies a strict inequality, only points strictly on one side of it.

What is a feasible region?

The solution region of a system of linear inequalities, restricted to the first quadrant, containing every feasible solution to the constraints.

Where does the optimal value of a linear objective function always occur in a feasible region?

At one of the corner points (vertices) of the feasible region, never at an interior point.



Long Questions & Answers

Explain the complete procedure for graphing a system of two linear inequalities in two variables.

What is the first step in graphing an inequality?

Write the associated (corresponding) equation by replacing the inequality symbol with an equal sign, then find its x- and y-intercepts to plot the line.

How is the boundary line drawn?

The line is drawn dashed if the inequality is strict ($<$ or $>$), or solid if the inequality is non-strict ($\leq$ or $\geq$), since a solid line indicates the boundary itself is part of the solution.

How is the correct half plane identified?

A test point not on the line — usually the origin — is substituted into the inequality; if it satisfies the inequality, that side is shaded, otherwise the opposite side is shaded.

How is the solution to a system of two inequalities found?

Each inequality's half plane is graphed on the same axes, and the overlapping (intersecting) region of both shaded half planes is the solution region for the whole system.

Describe how the feasible region is used to solve a linear programming problem, with an example.

What is a feasible region and how is it obtained?

A feasible region is the solution region of a system of problem-constraint inequalities restricted to the first quadrant ($x \geq 0$, $y \geq 0$), found by graphing and intersecting all the constraint half planes.



What is a corner point and why does it matter?

A corner point is a vertex of the feasible region where two boundary lines meet; the maximum or minimum value of a linear objective function is always found at one of these points, not inside the region.

What is the step-by-step procedure to find an optimal solution?

Graph the constraints to determine the feasible region, list all its corner points, then evaluate the objective function at each corner point.

How is the final answer determined from the evaluated values?

The corner point giving the largest value of the objective function is the maximizing solution, and the one giving the smallest value is the minimizing solution — as illustrated by $f(x,y)=2x+3y$ reaching its maximum of 12 at (0,4).

Multiple Choice Questions (MCQs)

Which of the following is a linear equation?

- A. $5x > 7$
- B. $4x - 2 < 1$
- C. $2x + 1 = 1$
- D. $4 = 1 + 3$

Answer: C. $2x + 1 = 1$

Explanation: $2x + 1 = 1$ is an equation (uses $=$) with the variable to the first power, making it linear.

The solution of $5x - 10 = 10$ is:

- A. 0
- B. 50
- C. 4
- D. -4

Answer: C. 4

Explanation: $5x = 20$, so $x = 4$.

If $7x + 4 < 6x + 6$, then x belongs to the interval:



- A. $(2, \infty)$
- B. $[2, \infty)$
- C. $(-\infty, 2)$
- D. $(-\infty, 2]$

Answer: C. $(-\infty, 2)$

Explanation: $7x - 6x < 6 - 4$ gives $x < 2$, so the interval is $(-\infty, 2)$.

A vertical line divides the plane into:

- A. left half plane
- B. right half plane
- C. full plane
- D. left and right half planes

Answer: D. left and right half planes

Explanation: A vertical line splits the coordinate plane into a left half plane and a right half plane.

The equation formed from a linear inequality is called its:

- A. cubic equation
- B. associated equation
- C. quadratic equation
- D. feasible region

Answer: B. associated equation

Explanation: Replacing the inequality symbol with an equal sign gives the associated (or corresponding) equation.

$3x + 4 < 0$ is a(n):

- A. equation
- B. inequality
- C. not inequality
- D. identity

Answer: B. inequality

Explanation: It uses the strict less-than symbol, making it an inequality, not an equation.

A corner point is also called a:

- A. code
- B. vertex
- C. curve



D. region

Answer: B. vertex

Explanation: Corner point and vertex are used interchangeably for a boundary intersection point of a feasible region.

The point (0, 0) is closest to satisfying which inequality (as a boundary/test case)?

A. $4x + 5y > 8$

B. $3x + y > 6$

C. $-2x + 3y < 0$

D. $x + y > 4$

Answer: C. $-2x + 3y < 0$

Explanation: Substituting (0, 0) into $-2x + 3y$ gives 0, right on the boundary of the inequality, making it the natural test-point case among the options — a reminder that the origin is always tried first as a test point unless the boundary line itself passes through it.

The solution region restricted to the first quadrant is called the:

A. objective region

B. feasible region

C. solution region

D. constraints region

Answer: B. feasible region

Explanation: This restricted region is specifically termed the feasible region.

A function that is to be maximized or minimized is called the:

A. solution function

B. objective function

C. feasible function

D. none of these

Answer: B. objective function

Explanation: Such a function is called the objective function in linear programming.



Quick Revision Summary

- A linear equation in one variable always has exactly one solution; solve by isolating the variable through balanced operations.
- Multiplying or dividing an inequality by a negative number reverses its direction — this is the single most common source of errors.
- A dashed boundary line means strict inequality ($<$ or $>$); a solid line means non-strict ($\leq$ or $\geq$).
- Use a test point, usually the origin, to determine which half plane satisfies an inequality.
- The solution of a system of inequalities is the intersection of all the individual half planes.
- The optimal value of a linear objective function always occurs at a corner point of the feasible region — evaluate the function at every corner point to find it.

Exam Tips

- Always check a solved equation by substituting the answer back into the original equation.
- When multiplying or dividing an inequality by a negative number, flip the inequality symbol immediately — don't wait until the end.
- Pick a test point that is clearly not on the boundary line; the origin works unless the line itself passes through $(0,0)$.
- Sketch all constraint lines on the same axes before shading, so the final feasible region is easy to see.
- List every corner point of the feasible region methodically before evaluating the objective function, so none are missed.
- In word problems, remember that $x \geq 0$ and $y \geq 0$ constraints restrict the feasible region to the first quadrant only.



Unit 6: Trigonometry

Trigonometry studies the relationships between the angles and sides of a triangle, most powerfully in the right-angled triangle, and it is one of the most widely applied branches of mathematics — used in physics, engineering, architecture, astronomy, surveying, and navigation. This unit begins with the language of angles themselves: how an angle is defined in standard position, how it is measured in degrees versus radians, and how to convert fluently between the two systems, including the degree-minute-second notation used in navigation and surveying.

From there, the unit builds the six trigonometric ratios — sine, cosine, tangent, and their reciprocals — from a right-angled triangle, derives the fundamental Pythagorean identities, and uses them to prove more complex trigonometric identities. The unit closes with genuinely practical applications: solving a right triangle when only some of its sides and angles are known, and using angles of elevation and depression to find heights and distances that would otherwise be dangerous or impossible to measure directly.

Learning Objectives

- Identify angles in standard position and express them in both degrees and radians.
- Convert between decimal degrees and degrees-minutes-seconds (DMS) notation, and between degrees and radians.
- Compute arc length and area of a circular sector using the radian measure of the central angle.
- Apply the Pythagoras theorem and define the sine, cosine, and tangent ratios (and their reciprocals) for an acute angle of a right triangle.
- Use the complementary angle relationships between trigonometric ratios.
- Prove trigonometric identities using the fundamental Pythagorean identities and ratio definitions.
- Recall and apply the exact trigonometric ratios of the special angles 0° , 30° , 45° , 60° , and 90° .



- Solve a right triangle given a sufficient combination of known sides and angles.
- Solve real-life problems in two dimensions involving angles of elevation and depression.

Key Concepts

6.1 Angles in Standard Position: Degrees and Radians

An angle is formed by two rays sharing a common endpoint (the vertex); it is in standard position when its vertex is at the origin and its initial side lies along the positive x-axis. An angle is positive if measured counterclockwise from the initial side, and negative if measured clockwise. Co-terminal angles share the same initial and terminal sides but differ by a multiple of 360° (or 2π radians).

A degree is $1/360$ of a full rotation, further divided into 60 minutes (') per degree and 60 seconds (") per minute. A radian is defined as the angle subtended at a circle's center by an arc equal in length to the radius; a full rotation equals 2π radians. The conversions are $1 \text{ rad} = 180^\circ/\pi$ and $1^\circ = \pi/180 \text{ rad}$.

Arc Length and Sector Area

For a circle of radius r and a central angle θ measured in radians, the arc length is $\ell = r\theta$ and the area of the sector is $A = (1/2)r^2\theta$. Both formulas follow directly from the fact that a full circle (angle 2π) has circumference $2\pi r$ and area πr^2 , so ℓ and A scale proportionally with $\theta/2\pi$.

6.2 Trigonometric Ratios of an Acute Angle

In a right triangle with angle θ , the side opposite θ is the perpendicular, the side adjacent to θ (not the hypotenuse) is the base, and the side opposite the right angle is the hypotenuse. The six ratios are $\sin \theta = \text{perpendicular/hypotenuse}$, $\cos \theta = \text{base/hypotenuse}$, $\tan \theta = \text{perpendicular/base}$, and their reciprocals $\text{cosec } \theta = 1/\sin \theta$, $\sec \theta = 1/\cos \theta$, $\cot \theta = 1/\tan \theta$. Also, $\tan \theta = \sin \theta/\cos \theta$ and $\cot \theta = \cos \theta/\sin \theta$.



For complementary angles (θ and $90^\circ - \theta$), each ratio equals the 'co-ratio' of the other angle: $\sin(90^\circ - \theta) = \cos \theta$, $\cos(90^\circ - \theta) = \sin \theta$, $\tan(90^\circ - \theta) = \cot \theta$, and similarly for the reciprocal ratios.

6.3 Fundamental Trigonometric Identities

The Pythagoras theorem $a^2 + b^2 = c^2$ gives rise to three Pythagorean identities by dividing through by c^2 , b^2 , or a^2 respectively: $\sin^2\theta + \cos^2\theta = 1$, $\tan^2\theta + 1 = \sec^2\theta$, and $1 + \cot^2\theta = \operatorname{cosec}^2\theta$. These three identities are the foundation for proving virtually every other trigonometric identity in this unit, typically by converting all ratios to sine and cosine and simplifying algebraically.

6.4 Trigonometric Ratios of Special Angles

Using a right isosceles triangle (square cut along its diagonal) gives the ratios for 45° : $\sin 45^\circ = \cos 45^\circ = 1/\sqrt{2}$, $\tan 45^\circ = 1$. Using an equilateral triangle bisected into two 30-60-90 triangles gives the ratios for 30° and 60° : $\sin 30^\circ = \cos 60^\circ = 1/2$, $\cos 30^\circ = \sin 60^\circ = \sqrt{3}/2$, $\tan 30^\circ = 1/\sqrt{3}$, $\tan 60^\circ = \sqrt{3}$. These special-angle values should be memorized, since they appear constantly in exercises and applications.

6.5 Solving a Right Triangle

Solving a triangle means finding all its unknown sides and angles from the ones already known. For a right triangle, knowing any two of the three sides, or one side and one acute angle, is enough to find everything else, using the trigonometric ratios and the fact that the two acute angles sum to 90° . Four typical cases arise: one side and one angle known, the hypotenuse and one angle known, two sides known (using Pythagoras and an inverse trig ratio), and one side plus the hypotenuse known.

6.6 Angle of Elevation and Angle of Depression

The angle of elevation is the angle between a horizontal line of sight and the line to an object above that horizontal — used, for instance, when looking up at the top of a building. The angle of depression is the angle between a horizontal line of sight and the line to an object below that horizontal — used when looking down from a height at an object on the ground. These angles let heights and



distances be calculated indirectly using the tangent ratio, without needing to physically measure a dangerous or inaccessible height.

Important Definitions

Angle in Standard Position

An angle whose vertex is at the origin of the coordinate plane and whose initial side lies along the positive x-axis.

Co-terminal Angles

Angles that share the same initial and terminal sides in standard position but differ in measure by a multiple of 360° (or 2π radians).

Radian

The angle subtended at the center of a circle by an arc whose length equals the circle's radius; a full rotation equals 2π radians.

Sine, Cosine, Tangent

The three primary trigonometric ratios of an acute angle θ in a right triangle: $\sin \theta = \text{perpendicular/hypotenuse}$, $\cos \theta = \text{base/hypotenuse}$, $\tan \theta = \text{perpendicular/base}$.

Pythagorean Identities

The three identities $\sin^2\theta + \cos^2\theta = 1$, $\tan^2\theta + 1 = \sec^2\theta$, and $1 + \cot^2\theta = \text{cosec}^2\theta$, derived from the Pythagoras theorem.

Solving a Triangle

The process of finding all unknown sides and angles of a triangle from a given sufficient set of known sides and angles.

Angle of Elevation

The angle between a horizontal line of sight and the line of sight to an object located above the horizontal.

Angle of Depression

The angle between a horizontal line of sight and the line of sight to an object located below the horizontal.

Sector

The region of a circle bounded by two radii and the arc between them.



Degree-Minute-Second (DMS)

A way of expressing an angle using degrees, minutes ($1^\circ = 60'$), and seconds ($1' = 60''$) for finer precision than decimal degrees alone.

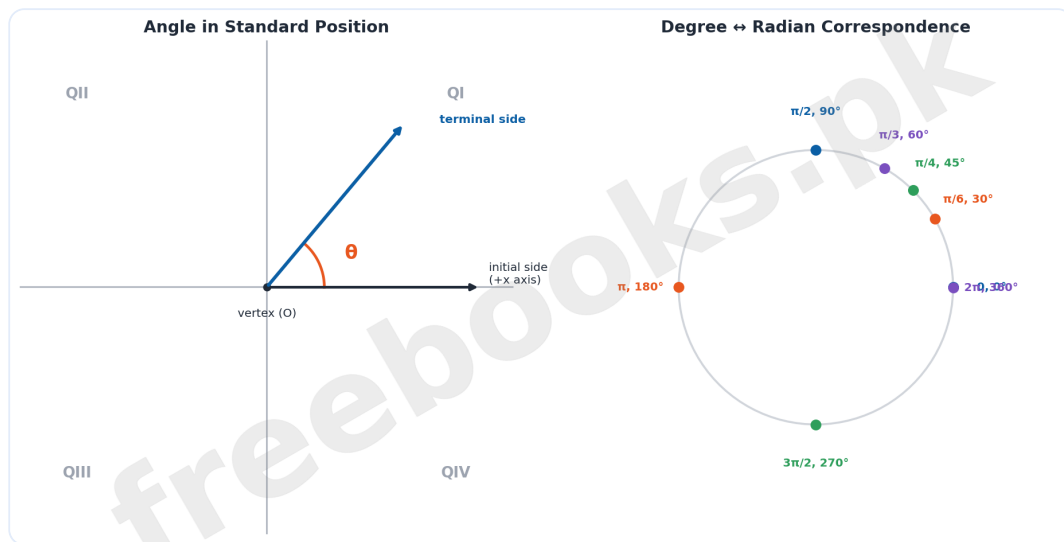
Key Facts and Relations

Topic	Key Fact / Relation
$1^\circ = \pi/180 \text{ rad} ; 1 \text{ rad} = 180^\circ/\pi$	Conversion between degrees and radians.
$\ell = r\theta$	Arc length of a sector, with θ in radians.
$A = (1/2)r^2\theta$	Area of a sector, with θ in radians.
$\sin \theta = a/c, \cos \theta = b/c, \tan \theta = a/b$	Primary trigonometric ratios for a right triangle with perpendicular a, base b, hypotenuse c.
$\operatorname{cosec} \theta = 1/\sin \theta, \sec \theta = 1/\cos \theta, \cot \theta = 1/\tan \theta$	Reciprocal trigonometric ratios.
$\sin^2\theta + \cos^2\theta = 1$	First Pythagorean identity.
$\tan^2\theta + 1 = \sec^2\theta$	Second Pythagorean identity.
$1 + \cot^2\theta = \operatorname{cosec}^2\theta$	Third Pythagorean identity.
$\sin(90^\circ - \theta) = \cos \theta ; \cos(90^\circ - \theta) = \sin \theta$	Complementary angle identities.
$\tan \theta = \text{height} / \text{distance}$ (elevation/depression problems)	Basic relation used to solve real-world height and distance problems.

Diagrams

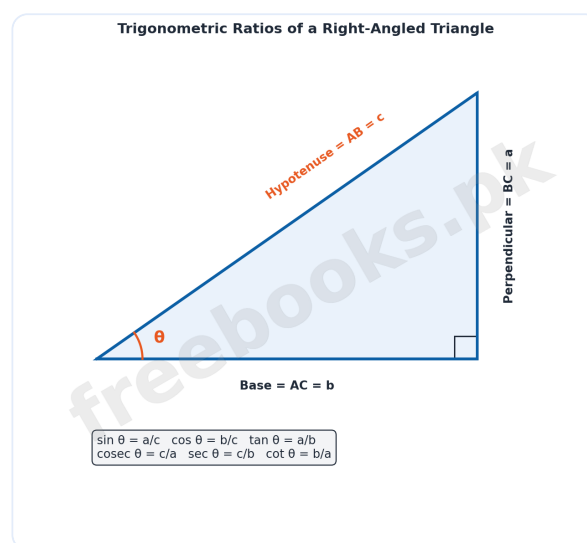
Angle in Standard Position and Degree-Radian Correspondence: Two-panel diagram showing an angle drawn in standard position with quadrants labeled, alongside a circle marking key angles in both degrees and radians





Angle in Standard Position and Degree-Radian Correspondence

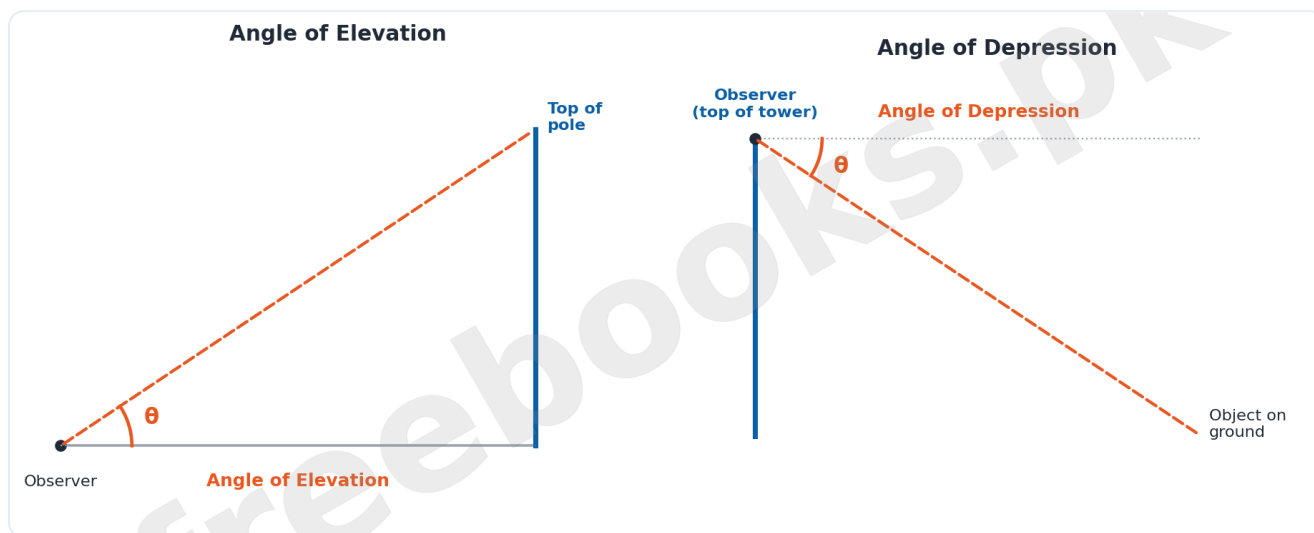
Trigonometric Ratios of a Right-Angled Triangle: A labeled right triangle showing the perpendicular, base, and hypotenuse along with all six trigonometric ratio definitions



Trigonometric Ratios of a Right-Angled Triangle

Angle of Elevation and Angle of Depression: Side-by-side diagrams illustrating the angle of elevation to the top of a pole and the angle of depression from a tower to a ground object





Angle of Elevation and Angle of Depression

Solved Examples

Example 1: Converting Decimal Degrees to Degrees, Minutes, and Seconds

Problem: Convert 73.12° to degrees, minutes, and seconds.

1. Keep the whole-number part as degrees: 73° .
2. Multiply the decimal part by 60 to get minutes: $0.12 \times 60 = 7.2'$, so 7 whole minutes.
3. Multiply the remaining decimal part of the minutes by 60 to get seconds: $0.2 \times 60 = 12''$.
4. Combine the results: $73^\circ 7' 12''$.

Example 2: Converting Between Radians and Degrees

Problem: Convert $5\pi/3$ rad to degrees, and convert 75° to radians.

1. To convert radians to degrees, multiply by $180^\circ/\pi$: $(5\pi/3) \times (180^\circ/\pi) = 300^\circ$.
2. To convert degrees to radians, multiply by $\pi/180$: $75 \times (\pi/180) = 5\pi/12$ rad, approximately 1.309 rad.



Example 3: Finding Arc Length and Sector Area

Problem: Find the arc length of a sector with radius 10 cm and central angle 60° , and the area of a sector with radius 8 cm and central angle 45° .

1. Convert 60° to radians: $60 \times \pi/180 = \pi/3$ rad.
2. Apply the arc length formula: $\ell = r\theta = 10 \times \pi/3 \approx 10.47$ cm.
3. Convert 45° to radians: $45 \times \pi/180 = \pi/4$ rad.
4. Apply the sector area formula: $A = (1/2)r^2\theta = (1/2)(64)(\pi/4) = 8\pi \approx 25.12$ cm².

Example 4: Proving a Trigonometric Identity Using Pythagorean Identities

Problem: Show that $\tan \theta + \cot \theta = \sec \theta \operatorname{cosec} \theta$.

1. Start from the left-hand side and write both terms using sine and cosine: $(\sin \theta / \cos \theta) + (\cos \theta / \sin \theta)$.
2. Combine over a common denominator: $(\sin^2 \theta + \cos^2 \theta) / (\sin \theta \cos \theta)$.
3. Apply the Pythagorean identity $\sin^2 \theta + \cos^2 \theta = 1$: result is $1 / (\sin \theta \cos \theta)$.
4. Split into reciprocals: $(1/\cos \theta)(1/\sin \theta) = \sec \theta \operatorname{cosec} \theta$, matching the right-hand side.

Example 5: Finding Remaining Trigonometric Ratios From One Given Ratio

Problem: If $\tan \theta = 3/4$ and θ lies in the first quadrant, find the remaining trigonometric ratios.

1. Interpret $\tan \theta = 3/4$ as perpendicular $a = 3$ and base $b = 4$.
2. Use Pythagoras theorem to find the hypotenuse: $c^2 = a^2 + b^2 = 9 + 16 = 25$, so $c = 5$.
3. Compute $\sin \theta = a/c = 3/5$ and $\cos \theta = b/c = 4/5$.



4. Compute the reciprocals: $\operatorname{cosec} \theta = 5/3$, $\sec \theta = 5/4$, $\cot \theta = 4/3$.

Example 6: Deriving the Special Angle Ratios for 30° and 60°

Problem: Find the exact trigonometric ratios of 30° and 60° using an equilateral triangle.

1. Start with an equilateral triangle of side 2 units and draw the perpendicular bisector from one vertex, splitting it into two right triangles.
2. Each right triangle has a base of 1 unit, hypotenuse of 2 units, and angles of 30° , 60° , 90° .
3. Apply Pythagoras theorem to find the remaining side: $x^2 = 2^2 - 1^2 = 3$, so $x = \sqrt{3}$.
4. Read off the ratios: $\sin 30^\circ = 1/2$, $\cos 30^\circ = \sqrt{3}/2$, $\tan 30^\circ = 1/\sqrt{3}$; $\sin 60^\circ = \sqrt{3}/2$, $\cos 60^\circ = 1/2$, $\tan 60^\circ = \sqrt{3}$.

Example 7: Solving a Right Triangle Given One Side and One Angle

Problem: Solve triangle ABC where $\angle B = 90^\circ$, $\angle A = 30^\circ$, and side $a = 2$ cm.

1. Find the third angle: $\angle C = \angle B - \angle A = 90^\circ - 30^\circ = 60^\circ$.
2. Use $\sin 30^\circ = a/b$ to find side b : $2/b = 1/2$, so $b = 4$ cm.
3. Use $\tan 30^\circ = a/c$ to find side c : $2/c = 1/\sqrt{3}$, so $c = 2\sqrt{3}$ cm.
4. State the complete solution: $\angle C = 60^\circ$, $b = 4$ cm, $c = 2\sqrt{3}$ cm.

Example 8: Real-World Application: Finding Height Using the Angle of Elevation

Problem: The angle of elevation of the top of a 40 m high pole is 60° from a point on the ground. Find the distance of the point from the foot of the pole.

1. Let x be the distance from the observation point to the foot of the pole.
2. Set up the tangent ratio: $\tan 60^\circ = (\text{height})/(\text{distance}) = 40/x$.



3. Substitute $\tan 60^\circ = \sqrt{3}$: $\sqrt{3} = 40/x$, so $x = 40/\sqrt{3}$.
4. Simplify: $x \approx 23.09$ m, the distance of the point from the foot of the pole.

Short Questions & Answers

What are the two conditions for an angle to be in standard position?

Its vertex must be at the origin of the coordinate plane, and its initial side must lie along the positive x-axis.

What defines a radian?

The angle subtended at the center of a circle by an arc whose length equals the circle's radius.

What are co-terminal angles?

Angles that share the same initial and terminal sides in standard position but whose measures differ by a multiple of 360° (or 2π radians).

In a right triangle, how is the tangent ratio related to sine and cosine?

$\tan \theta = \sin \theta / \cos \theta$, since dividing the perpendicular and base both by the hypotenuse gives this relationship directly.

State the three Pythagorean identities.

$\sin^2\theta + \cos^2\theta = 1$, $\tan^2\theta + 1 = \sec^2\theta$, and $1 + \cot^2\theta = \operatorname{cosec}^2\theta$.

What is the difference between the angle of elevation and the angle of depression?

The angle of elevation is measured upward from the horizontal to an object above, while the angle of depression is measured downward from the horizontal to an object below.

What information is needed to fully solve a right triangle?

At least two of its six elements (three sides and three angles) must be known, including at least one side, since the right angle is already known.



Long Questions & Answers

Explain how the fundamental trigonometric ratios and their reciprocals are defined, and how the Pythagorean identities follow from them.

How are the six trigonometric ratios defined in a right triangle?

For an acute angle θ , $\sin \theta = \text{perpendicular/hypotenuse}$, $\cos \theta = \text{base/hypotenuse}$, $\tan \theta = \text{perpendicular/base}$, and their reciprocals are $\operatorname{cosec} \theta = \text{hypotenuse/perpendicular}$, $\sec \theta = \text{hypotenuse/base}$, $\cot \theta = \text{base/perpendicular}$.

How is $\tan \theta$ related to $\sin \theta$ and $\cos \theta$?

Dividing both the perpendicular and base by the hypotenuse in $\tan \theta = \text{perpendicular/base}$ shows that $\tan \theta = \sin \theta / \cos \theta$, and similarly $\cot \theta = \cos \theta / \sin \theta$.

How is the first Pythagorean identity derived?

Starting from $a^2 + b^2 = c^2$ and dividing every term by c^2 gives $(a/c)^2 + (b/c)^2 = 1$, which is $\sin^2 \theta + \cos^2 \theta = 1$.

How are the other two Pythagorean identities derived?

Dividing $a^2 + b^2 = c^2$ by b^2 gives $\tan^2 \theta + 1 = \sec^2 \theta$, and dividing by a^2 gives $1 + \cot^2 \theta = \operatorname{cosec}^2 \theta$ — the same theorem, viewed through two different divisions.

Describe the procedure for solving real-world problems involving angles of elevation and depression, with worked reasoning.

What is the first step in an elevation or depression problem?

Sketch a right triangle representing the situation, clearly marking the horizontal line, the known height or distance, and the given angle as either an angle of elevation or depression.



Which trigonometric ratio is most commonly used, and why?

The tangent ratio is used most often, since it directly relates the vertical height (perpendicular) to the horizontal distance (base) without needing the hypotenuse.

How is the unknown value calculated once the triangle is set up?

The known angle and known side are substituted into the tangent equation, and the equation is solved algebraically for the unknown height or distance.

How does this apply to a real elevation example?

For a 40 m pole with an angle of elevation of 60° , $\tan 60^\circ = 40/x$ gives $x = 40/\sqrt{3} \approx 23.09$ m — finding the horizontal distance without ever measuring it directly.

Multiple Choice Questions (MCQs)

The value of $\tan^{-1}2$ in radians is approximately:

- A. $\pi/2$
- B. $3\pi/2$
- C. 1.11π
- D. 1.11

Answer: D. 1.11

Explanation: $\tan^{-1}2 \approx 63.43^\circ$, which in radians is approximately 1.11 (not multiplied by π).

In a right triangle with hypotenuse 13 units and one angle 30° , the opposite side length is:

- A. 6.5 units
- B. 7.5 units
- C. 6 units
- D. 5 units

Answer: A. 6.5 units

Explanation: $\sin 30^\circ = \text{opposite/hypotenuse} = 1/2$, so opposite = $13 \times 1/2 = 6.5$ units.



A person 50 m from a building sees its top at an angle of elevation of 45° . The building's height is:

- A. 50 m
- B. 25 m
- C. 35 m
- D. 70 m

Answer: A. 50 m

Explanation: $\tan 45^\circ = \text{height}/50 = 1$, so height = 50 m.

$\sec^2\theta - \tan^2\theta$ equals:

- A. $\sin^2\theta$
- B. 1
- C. $\cos^2\theta$
- D. $\cot^2\theta$

Answer: B. 1

Explanation: This is a rearrangement of the identity $\tan^2\theta + 1 = \sec^2\theta$, giving $\sec^2\theta - \tan^2\theta = 1$.

If $\sin \theta = 3/5$ and θ is acute, $\cos^2\theta$ equals:

- A. $7/25$
- B. $24/25$
- C. $16/25$
- D. $4/25$

Answer: C. $16/25$

Explanation: $\cos^2\theta = 1 - \sin^2\theta = 1 - 9/25 = 16/25$.

$5\pi/24$ rad equals how many degrees?

- A. 30°
- B. 37.5°
- C. 45°
- D. 52.5°

Answer: B. 37.5°

Explanation: $5\pi/24 \times 180/\pi = 900/24 = 37.5^\circ$.

Which of the following is a valid identity?

- A. $\cos(\pi/2 - \theta) = \sin \theta$
- B. $\cos(\pi/2 - \theta) = \cos \theta$
- C. $\cos(\pi/2 - \theta) = \sec \theta$



D. $\cos(\pi/2 - \theta) = \operatorname{cosec} \theta$

Answer: A. $\cos(\pi/2 - \theta) = \sin \theta$

Explanation: This is the standard complementary angle identity: $\cos(90^\circ - \theta) = \sin \theta$.

sin 60° equals:

A. 1

B. $1/2$

C. $\sqrt{(3^2)}$

D. $\sqrt{3}/2$

Answer: D. $\sqrt{3}/2$

Explanation: From the special-angle table, $\sin 60^\circ = \sqrt{3}/2$.

A radian is defined using:

A. the diameter of a circle

B. an arc equal in length to the radius

C. half the circumference

D. one-fourth of a full rotation

Answer: B. an arc equal in length to the radius

Explanation: A radian is the angle subtended by an arc whose length equals the circle's radius.

The angle between a horizontal line of sight and the line to an object below is called the:

A. angle of elevation

B. angle of depression

C. co-terminal angle

D. reference angle

Answer: B. angle of depression

Explanation: This downward-measured angle is specifically called the angle of depression.

Quick Revision Summary

- An angle in standard position has its vertex at the origin and initial side along the positive x-axis; positive angles rotate counterclockwise, negative angles clockwise.



- Convert degrees to radians by multiplying by $\pi/180$, and radians to degrees by multiplying by $180/\pi$.
- Arc length $l = r\theta$ and sector area $A = (1/2)r^2\theta$, always using θ in radians.
- The six trigonometric ratios are defined from a right triangle's perpendicular, base, and hypotenuse; $\tan \theta = \sin \theta / \cos \theta$.
- The three Pythagorean identities — $\sin^2\theta + \cos^2\theta = 1$, $\tan^2\theta + 1 = \sec^2\theta$, $1 + \cot^2\theta = \operatorname{cosec}^2\theta$ — are the key tool for proving other identities.
- Memorize the exact ratios for 0° , 30° , 45° , 60° , 90° — they appear constantly throughout the unit and beyond.
- In elevation/depression problems, sketch the triangle first and use the tangent ratio to relate height and horizontal distance.

Exam Tips

- Always double-check whether an angle is given in degrees or radians before applying any formula — mixing the two is the most common error.
- When proving an identity, convert every ratio to sine and cosine first; this almost always simplifies the algebra.
- Memorize the 30-45-60 special angle table rather than re-deriving it every time — it saves significant time in exams.
- In a right triangle problem, always identify which side is the perpendicular, base, and hypotenuse relative to the given angle before choosing a ratio.
- Draw a clear diagram for every elevation or depression word problem — most errors come from misidentifying which angle is given.
- Remember co-terminal angles differ by exactly 360° (or 2π) — never any other value.



Unit 7: Coordinate Geometry

Coordinate geometry, also called analytic geometry, links algebra and geometry by describing every point in a plane with an ordered pair of numbers. This idea, introduced by René Descartes, means that geometric problems — distances, midpoints, and the equations of lines — can all be solved using algebra instead of relying purely on diagrams and constructions. This unit builds that toolkit from the ground up: locating points in the coordinate plane, deriving the distance and midpoint formulas from the Pythagoras theorem, and defining the slope of a line as a measure of its steepness.

The unit then develops six different standard forms for the equation of a straight line — slope-intercept, point-slope, two-point, intercept, symmetric, and normal form — showing that they are all different ways of expressing the same underlying relationship, and that any general linear equation $ax + by + c = 0$ can be converted freely between them. The unit closes with genuinely practical applications: using these formulas for physical measurements, real-world distances, aviation headings, and even latitude-longitude midpoints.

Learning Objectives

- Locate points in the coordinate plane using ordered pairs and identify the quadrant of a given point.
- Derive and apply the distance formula to find the distance between two points.
- Derive and apply the midpoint formula to find the midpoint of a line segment.
- Find the gradient (slope) of a straight line joining two given points.
- Determine the conditions for two lines to be parallel or perpendicular using their slopes.
- Find the equation of a straight line in slope-intercept, point-slope, two-point, intercept, symmetric, and normal form.
- Show that a linear equation in two variables always represents a straight line, and convert the general form into any of the standard forms.



- Apply distance, midpoint, and slope formulas to real-life situations such as physical measurements, navigation, and map reading.

Key Concepts

7.1 The Coordinate Plane

Two mutually perpendicular number lines — the horizontal x-axis and vertical y-axis — meet at the origin O and divide the plane into four quadrants. Any point P in the plane is located by an ordered pair (x, y), where x (the abscissa) is its signed horizontal distance from the y-axis and y (the ordinate) is its signed vertical distance from the x-axis. Quadrant I has $x > 0, y > 0$; Quadrant II has $x < 0, y > 0$; Quadrant III has $x < 0, y < 0$; Quadrant IV has $x > 0, y < 0$.

7.1.1 - 7.1.2 Distance and Midpoint Formulas

For two points $A(x_1, y_1)$ and $B(x_2, y_2)$, drawing a right triangle with legs parallel to the axes and applying the Pythagoras theorem gives the distance formula $d = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2]}$. Distance is always taken as non-negative, and either point may be labeled first without changing the result.

The midpoint M of the segment AB is found by averaging the x-coordinates and averaging the y-coordinates separately: $M(x, y) = ((x_1 + x_2)/2, (y_1 + y_2)/2)$. Both formulas are used constantly throughout the unit, including to test whether three points form a right triangle, an isosceles triangle, or a parallelogram.

7.2 Slope (Gradient) of a Line

The inclination α of a line is the angle ($0^\circ < \alpha < 180^\circ$) measured counterclockwise from the positive x-axis to the line. The slope m is defined as $m = \tan \alpha$, and equivalently, for two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ on a non-vertical line, $m = (y_2 - y_1)/(x_2 - x_1)$. A horizontal line has slope 0; a vertical line has undefined slope. Two lines are parallel exactly when their slopes are equal ($m_1 = m_2$), and perpendicular exactly when the product of their slopes is -1 ($m_1 m_2 = -1$).



Three points A, B, C are collinear exactly when the slope of AB equals the slope of BC — this is a common technique for verifying collinearity or classifying a triangle as right-angled without directly measuring angles.

7.2.4 Six Standard Forms of a Line's Equation

Slope-intercept form: $y = mx + c$, where m is the slope and c is the y-intercept.
Point-slope form: $y - y_1 = m(x - x_1)$, for a line through a known point with known slope.
Two-point form: $y - y_1 = [(y_2 - y_1)/(x_2 - x_1)](x - x_1)$, for a line through two known points.

Intercept form: $x/a + y/b = 1$, where a and b are the x- and y-intercepts.
Symmetric form: $(x - x_1)/\cos \alpha = (y - y_1)/\sin \alpha = r$, expressed using the line's inclination.
Normal form: $x \cos \alpha + y \sin \alpha = p$, where p is the perpendicular distance from the origin to the line and α is the inclination of that perpendicular.

7.2.5 - 7.2.6 The General Linear Equation

Every linear equation $ax + by + c = 0$ (with a, b not both zero) represents a straight line. If $b = 0$, the line is vertical ($x = -c/a$); if $a = 0$, the line is horizontal ($y = -c/b$); if neither is zero, the equation reduces to slope-intercept form with $m = -a/b$ and y-intercept $-c/b$. Any such general equation can be systematically converted into any of the six standard forms by algebraic rearrangement.

7.3 Real-World Applications

Coordinate geometry formulas apply directly to physical and navigational problems: distances between towns or delivery points on a map, midpoints of bridges or pathways, side lengths of triangular plots of land, perimeters of rectangular fields, and even the heading angle a pilot should take (found using the arctangent of the slope between two coordinates). Latitude and longitude can also be averaged using the midpoint formula to find the midpoint of a journey.

Important Definitions

Coordinate Plane



A plane formed by two mutually perpendicular number lines, the x-axis and y-axis, intersecting at the origin, used to locate points via ordered pairs.

Abscissa and Ordinate

The x-coordinate (abscissa) and y-coordinate (ordinate) of a point, together forming its ordered pair (x, y) .

Distance Formula

The formula $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ for the straight-line distance between two points $A(x_1, y_1)$ and $B(x_2, y_2)$, derived from the Pythagoras theorem.

Midpoint Formula

The formula $M = ((x_1 + x_2)/2, (y_1 + y_2)/2)$ giving the point exactly halfway between $A(x_1, y_1)$ and $B(x_2, y_2)$.

Inclination of a Line

The angle α ($0^\circ < \alpha < 180^\circ$) measured counterclockwise from the positive x-axis to a non-horizontal line.

Slope (Gradient)

The measure $m = \tan \alpha$ of a line's steepness, equivalently computed as $m = (y_2 - y_1)/(x_2 - x_1)$ for two points on the line.

x-intercept and y-intercept

The x-coordinate where a line crosses the x-axis (y-intercept: the y-coordinate where it crosses the y-axis).

General Equation of a Line

The form $ax + by + c = 0$, where a and b are not both zero, which always represents some straight line.

Normal Form of a Line

The equation $x \cos \alpha + y \sin \alpha = p$, where p is the perpendicular distance from the origin to the line and α is the inclination of that perpendicular.

Collinear Points

Points that lie on the same straight line; three points are collinear exactly when the slope between the first pair equals the slope between the second pair.



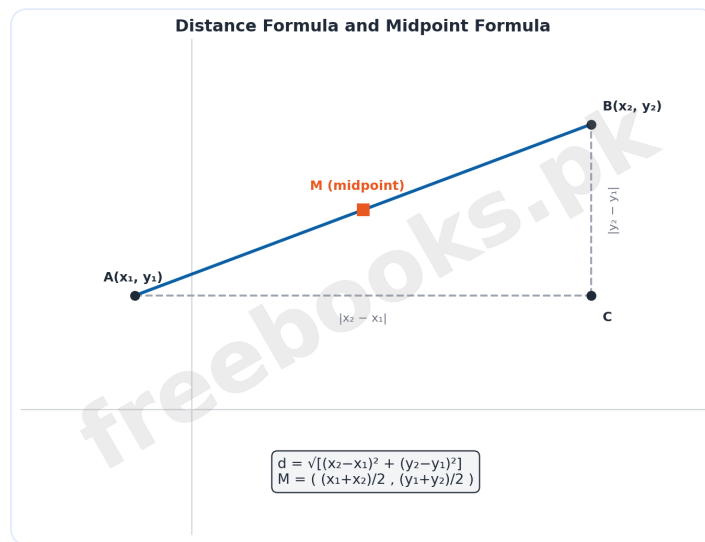
Key Facts and Relations

Topic	Key Fact / Relation
$d = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2]}$	Distance formula between two points.
$M(x,y) = ((x_1 + x_2)/2, (y_1 + y_2)/2)$	Midpoint formula for a line segment.
$m = (y_2 - y_1)/(x_2 - x_1) = \tan \alpha$	Slope of a line through two points, or via its inclination.
$m_1 = m_2$ (parallel) ; $m_1 m_2 = -1$ (perpendicular)	Conditions for two lines to be parallel or perpendicular.
$y = mx + c$	Slope-intercept form of a line's equation.
$y - y_1 = m(x - x_1)$	Point-slope form of a line's equation.
$x/a + y/b = 1$	Intercept form of a line's equation, with x-intercept a and y-intercept b.
$x \cos \alpha + y \sin \alpha = p$	Normal form of a line's equation.
$ax + by + c = 0$	General form of the equation of a straight line.

Diagrams

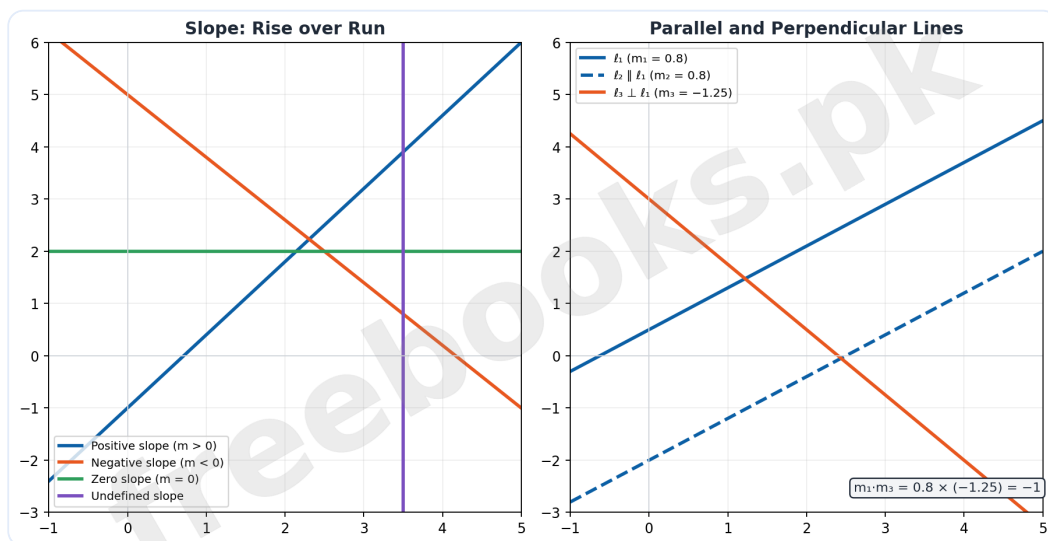
Distance Formula and Midpoint Formula: A coordinate-plane diagram showing two points joined by a right triangle to derive the distance formula, with the midpoint marked between them





Distance Formula and Midpoint Formula

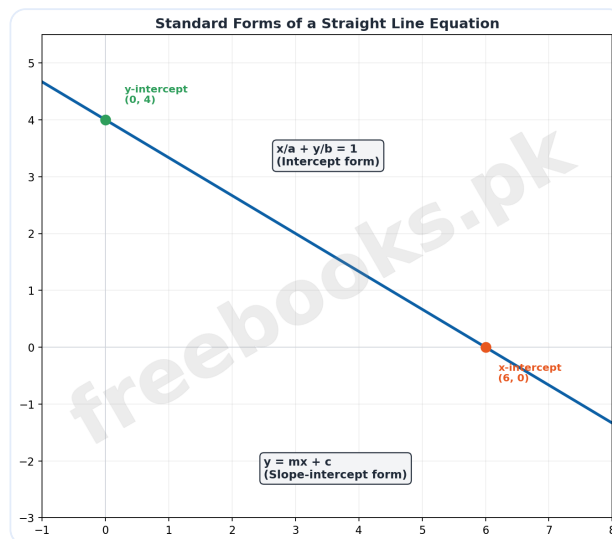
Slope: Rise Over Run, and Parallel/Perpendicular Lines: Two-panel diagram showing lines of positive, negative, zero, and undefined slope, and a second panel showing parallel and perpendicular line pairs with their slope relationship



Slope: Rise Over Run, and Parallel/Perpendicular Lines

Standard Forms of a Straight Line Equation: A graphed line showing its x-intercept and y-intercept, annotated with the intercept form and slope-intercept form of its equation





Standard Forms of a Straight Line Equation

Solved Examples

Example 1: Finding the Distance Between Two Points

Problem: Find the distance between the points C(−4, −2) and D(0, 9).

1. Apply the distance formula: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
2. Substitute the coordinates: $d = \sqrt{(0 - (-4))^2 + (9 - (-2))^2} = \sqrt{4^2 + 11^2}$.
3. Simplify: $d = \sqrt{16 + 121} = \sqrt{137}$.
4. State the result: the distance CD is $\sqrt{137}$ units.

Example 2: Showing Three Points Form a Right Triangle

Problem: Show that A(−1, 2), B(7, 5), and C(2, −6) are vertices of a right triangle.

1. Compute each side length using the distance formula: $|AB| = \sqrt{73}$, $|BC| = \sqrt{146}$, $|CA| = \sqrt{73}$.
2. Check the Pythagoras relationship: does the sum of two smaller squared sides equal the largest squared side?
3. Verify: $|CA|^2 + |AB|^2 = 73 + 73 = 146 = |BC|^2$, so the Pythagoras theorem holds.



4. Conclude that triangle ABC is right-angled, with the right angle at vertex A.

Example 3: Finding the Midpoint of a Line Segment

Problem: Find the midpoint of the segment joining A(2, 3) and B(8, 7).

1. Apply the midpoint formula: $M = ((x_1+x_2)/2, (y_1+y_2)/2)$.
2. Substitute the coordinates: $M = ((2+8)/2, (3+7)/2)$.
3. Simplify: $M = (10/2, 10/2) = (5, 5)$.

Example 4: Showing Three Points Are Collinear Using Slope

Problem: Show that A(-3, 6), B(3, 2), and C(6, 0) are collinear.

1. Compute the slope of AB: $(2-6)/(3-(-3)) = -4/6 = -2/3$.
2. Compute the slope of BC: $(0-2)/(6-3) = -2/3$.
3. Compare: slope of AB equals slope of BC, both equal to $-2/3$.
4. Since the slopes are equal and B is a shared point, conclude A, B, and C are collinear.

Example 5: Finding a Line's Equation Using Point-Slope Form

Problem: Find the equation of the line through (5, 1) that is parallel to the line through (0, -1) and (7, -15).

1. Find the slope of the given line: $m = (-15-(-1))/(7-0) = -14/7 = -2$.
2. Since parallel lines share the same slope, the required line also has slope -2 .
3. Apply point-slope form with point (5, 1): $y - 1 = -2(x - 5)$.
4. Simplify to general form: $y = -2x + 11$, or $2x + y - 11 = 0$.



Example 6: Converting a General Equation to Slope-Intercept and Intercept Form

Problem: Transform $5x - 12y + 39 = 0$ into slope-intercept form and intercept form.

1. For slope-intercept form, isolate y : $12y = 5x + 39$, so $y = (5/12)x + 39/12$, giving $m = 5/12$.
2. For intercept form, rewrite as $5x - 12y = -39$.
3. Divide through by -39 : $5x/(-39) + 12y/39 = 1$, i.e., $x/(-39/5) + y/(39/12) = 1$.
4. Read off the intercepts: x -intercept $= -39/5$, y -intercept $= 39/12$.

Example 7: Real-World Application: Finding a Bridge's Center Using the Midpoint Formula

Problem: An engineer is building a bridge between points $(2, 5)$ and $(8, 9)$ on a riverbank. Find the coordinates of the bridge's center.

1. Apply the midpoint formula to the two given points: $M = ((2+8)/2, (5+9)/2)$.
2. Simplify: $M = (10/2, 14/2) = (5, 7)$.
3. State the result: the center of the bridge is located at $(5, 7)$.

Example 8: Real-World Application: Finding an Aviation Heading Angle Using Slope

Problem: A pilot travels from city A(50, 60) to city B(120, 150). Find the heading angle relative to the east direction.

1. Compute the slope between the two cities: $m = (150-60)/(120-50) = 90/70 = 9/7$.
2. Relate the slope to the heading angle: $\tan \theta = m = 9/7$.
3. Solve for θ : $\theta = \tan^{-1}(9/7) = \tan^{-1}(1.2857) \approx 52.13^\circ$.



4. State the conclusion: the plane should take a heading angle of approximately 52.13° north of east.

Short Questions & Answers

What is the abscissa and ordinate of a point (x, y)?

The abscissa is the x-coordinate (horizontal position) and the ordinate is the y-coordinate (vertical position) of the point.

How is the distance formula derived?

By drawing a right triangle between two points with legs parallel to the axes and applying the Pythagoras theorem to the horizontal and vertical differences.

What does it mean for a line to have zero slope versus undefined slope?

A horizontal line has zero slope ($\alpha = 0^\circ$); a vertical line has undefined slope ($\alpha = 90^\circ$, where tangent is undefined).

What condition on slopes indicates that two lines are perpendicular?

The product of their slopes equals -1 , i.e., $m_1 m_2 = -1$.

What is the slope-intercept form of a line's equation?

$y = mx + c$, where m is the slope and c is the y-intercept.

What does the general equation $ax + by + c = 0$ represent when $a = 0$?

It reduces to $y = -c/b$, a horizontal line parallel to the x-axis.

How can three points be tested for collinearity using slope?

By checking whether the slope between the first pair of points equals the slope between the second pair; if equal, the points are collinear.



Long Questions & Answers

Explain how the distance and midpoint formulas are derived and used to classify triangles and quadrilaterals.

How is the distance formula derived?

For points $A(x_1, y_1)$ and $B(x_2, y_2)$, a right triangle is formed with horizontal leg $|x_2 - x_1|$ and vertical leg $|y_2 - y_1|$; applying the Pythagoras theorem gives $d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$, so $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

How is the midpoint formula derived?

The midpoint's x-coordinate is the average of the two x-coordinates, and its y-coordinate is the average of the two y-coordinates, giving $M = ((x_1 + x_2)/2, (y_1 + y_2)/2)$.

How is the distance formula used to identify a right triangle?

The three side lengths are computed with the distance formula, and the Pythagoras relationship (sum of squares of two sides equals the square of the third) is checked to confirm a right angle.

How are these formulas used to test a quadrilateral?

Side lengths and diagonal midpoints are computed; equal opposite sides or a shared diagonal midpoint can confirm shapes like parallelograms, following the same coordinate-based reasoning throughout.

Describe the six standard forms of a line's equation and how a general linear equation is converted between them.

What are the slope-intercept and point-slope forms?

Slope-intercept form is $y = mx + c$, using the slope and y-intercept; point-slope form is $y - y_1 = m(x - x_1)$, using the slope and any one known point on the line.



What are the two-point and intercept forms?

Two-point form is $y - y_1 = [(y_2 - y_1)/(x_2 - x_1)](x - x_1)$, built directly from two known points; intercept form is $x/a + y/b = 1$, using the x- and y-intercepts a and b.

What are the symmetric and normal forms?

Symmetric form is $(x - x_1)/\cos \alpha = (y - y_1)/\sin \alpha = r$, built from the inclination α ; normal form is $x \cos \alpha + y \sin \alpha = p$, using the perpendicular distance p from the origin and the inclination α of that perpendicular.

How is a general equation $ax + by + c = 0$ converted between these forms?

The equation is algebraically rearranged: isolating y gives slope-intercept form; picking a point on the line gives point-slope or two-point form; dividing by $-c$ gives intercept form; dividing by $\pm\sqrt{a^2 + b^2}$ gives normal form, choosing the sign so the right-hand side is positive.

Multiple Choice Questions (MCQs)

The equation of a straight line in slope-intercept form is written as:

- A. $y = m(x + c)$
- B. $y - y_1 = m(x - x_1)$
- C. $y = c + mx$
- D. $ax + by + c = 0$

Answer: C. $y = c + mx$

Explanation: Slope-intercept form is $y = mx + c$, which can be equivalently written $y = c + mx$.

The gradients of two parallel lines are:

- A. equal
- B. zero
- C. negative reciprocals of each other
- D. always undefined

Answer: A. equal

Explanation: Two lines are parallel exactly when their slopes are equal, $m_1 = m_2$.



If the product of the gradients of two lines is -1 , then the lines are:

- A. Parallel
- B. perpendicular
- C. Collinear
- D. coincident

Answer: B. perpendicular

Explanation: $m_1m_2 = -1$ is precisely the condition for two lines to be perpendicular.

The distance between the points P(1, 2) and Q(4, 6) is:

- A. 5
- B. 6
- C. $\sqrt{13}$
- D. 4

Answer: A. 5

Explanation: $d = \sqrt{(4-1)^2 + (6-2)^2} = \sqrt{9+16} = \sqrt{25} = 5$.

The midpoint of a line segment with endpoints $(-2, 4)$ and $(6, -2)$ is:

- A. (4, 2)
- B. (2, 1)
- C. (1, 1)
- D. (0, 0)

Answer: B. (2, 1)

Explanation: $M = ((-2+6)/2, (4-2)/2) = (4/2, 2/2) = (2, 1)$.

A line passing through points (1, 2) and (4, 5) is:

- A. $y = x + 1$
- B. $y = 2x + 3$
- C. $y = 3x - 2$
- D. $y = x + 2$

Answer: A. $y = x + 1$

Explanation: Slope = $(5-2)/(4-1) = 1$; using point-slope form: $y - 2 = 1(x - 1)$, so $y = x + 1$.

The equation of a line in point-slope form is:

- A. $y = m(x + c)$
- B. $y - y_1 = m(x - x_1)$
- C. $y = c + mx$



D. $ax + by + c = 0$

Answer: B. $y - y_1 = m(x - x_1)$

Explanation: Point-slope form is precisely $y - y_1 = m(x - x_1)$.

$2x + 3y - 6 = 0$ in slope-intercept form is:

A. $y = (-2/3)x + 2$

B. $y = (2/3)x - 2$

C. $y = (2/3)x + 1$

D. $y = (-2/3)x - 2$

Answer: A. $y = (-2/3)x + 2$

Explanation: $3y = -2x + 6$, so $y = (-2/3)x + 2$.

The equation of a line in symmetric form is:

A. $x/a + y/b = 1$

B. $(x-x_1)/1 + (y-y_1)/m = (z-z_1)/1$

C. $(x-x_1)/\cos \alpha = (y-y_1)/\sin \alpha = r$

D. $y - y_1 = m(x - x_1)$

Answer: C. $(x-x_1)/\cos \alpha = (y-y_1)/\sin \alpha = r$

Explanation: Symmetric form uses the inclination α : $(x-x_1)/\cos \alpha = (y-y_1)/\sin \alpha = r$.

The equation of a line in normal form is:

A. $y = mx + c$

B. $x/a + y/b = 1$

C. $(x-x_1)/\cos \alpha = (y-y_1)/\sin \alpha$

D. $x \cos \alpha + y \sin \alpha = p$

Answer: D. $x \cos \alpha + y \sin \alpha = p$

Explanation: Normal form is $x \cos \alpha + y \sin \alpha = p$, where p is the perpendicular distance from the origin.

Quick Revision Summary

- The distance formula $d = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$ comes directly from the Pythagoras theorem applied to a right triangle between two points.
- The midpoint formula averages the x-coordinates and y-coordinates separately: $M = ((x_1+x_2)/2, (y_1+y_2)/2)$.



- Slope $m = (y_2 - y_1)/(x_2 - x_1) = \tan \alpha$; parallel lines have equal slopes, perpendicular lines have slopes multiplying to -1 .
- There are six standard forms of a line's equation — slope-intercept, point-slope, two-point, intercept, symmetric, and normal — all describing the same line differently.
- Every linear equation $ax + by + c = 0$ (a, b not both zero) represents some straight line, and can be converted into any standard form.
- Real-world distance, midpoint, and slope problems (maps, navigation, latitude/longitude) use exactly the same formulas as pure coordinate geometry problems.

Exam Tips

- Always subtract coordinates in the same order in both the x and y terms of the distance formula to avoid sign errors.
- When testing collinearity, compute slopes between overlapping pairs of points (not all three independently) so a shared point can cancel out.
- Memorize the parallel condition ($m_1 = m_2$) and perpendicular condition ($m_1 m_2 = -1$) — they appear in nearly every exercise in this unit.
- When converting a general equation to a standard form, isolate y first for slope-intercept form — most other forms follow easily from there.
- Sketch a quick graph before solving a word problem — visualizing intercepts and slope prevents sign and setup errors.
- For normal form, always double-check the sign of the radical so the right-hand side p comes out positive.



Unit 8: Logic

Logic is the systematic study of reasoning — it gives mathematics a precise language for deciding whether a statement is true, combining statements into new ones, and building airtight chains of argument from accepted facts to new conclusions. This unit introduces the mathematical statement as the basic unit of logic, then builds up the five standard logical operators (negation, conjunction, disjunction, conditional, and biconditional) along with the truth tables that define exactly when each combined statement is true or false.

From there, the unit explores the deep relationships between a conditional and its converse, inverse, and contrapositive, and introduces the vocabulary mathematicians use to classify claims: axioms (accepted without proof), theorems (proved true), and conjectures (believed true but unproven). The unit closes by teaching the practical skill of writing a deductive proof — a step-by-step algebraic argument showing that one side of an equation equals the other, justifying every step with a known law or property.

Learning Objectives

- Understand what a mathematical statement is and what it means for a statement to be true or false.
- Differentiate between an axiom, a conjecture, and a theorem, with examples of each.
- Use the logical operators negation, conjunction, disjunction, conditional, and biconditional, and construct their truth tables.
- Form the converse, inverse, and contrapositive of a given conditional statement and understand the equivalences between them.
- Understand the difference between inductive and deductive reasoning.
- Formulate simple deductive proofs, including algebraic proofs that show the left-hand side of an equation equals the right-hand side.



Key Concepts

8.1 Mathematical Statements

A statement is a sentence or mathematical expression that is either true or false, but not both. Statements are the fundamental building blocks of logic — for example, ' $x^m \cdot x^n = x^{(m+n)}$ ' and 'the sum of two odd integers is an even integer' are both true mathematical statements, while ' $3 + 4 = 8$ ' is a false one.

Inductive reasoning draws general conclusions from repeated observations or experiments (common in natural sciences), while deductive reasoning draws conclusions from premises already accepted as true, guaranteeing the conclusion is true if the premises are true. Mathematics relies primarily on deductive reasoning to build reliable, provable knowledge.

8.1.1 - 8.1.2 Logical Operators and Truth Tables

Negation ($\sim p$, 'not p') reverses a statement's truth value. Conjunction ($p \wedge q$, 'p and q') is true only when both p and q are true. Disjunction ($p \vee q$, 'p or q') is true when at least one of p, q is true, and false only when both are false.

The conditional ($p \rightarrow q$, 'if p then q') is false only in the single case where p is true and q is false — in every other combination it is considered true. The biconditional ($p \leftrightarrow q$, 'p if and only if q') is true exactly when p and q share the same truth value, whether both true or both false.

Converse, Inverse, and Contrapositive

For a given conditional $p \rightarrow q$: the converse is $q \rightarrow p$ (swapping the two statements), the inverse is $\sim p \rightarrow \sim q$ (negating both statements), and the contrapositive is $\sim q \rightarrow \sim p$ (negating both statements and swapping their order). A conditional and its contrapositive always share the same truth value (they are logically equivalent), and likewise the converse and the inverse are always equivalent to each other — this is why a theorem can sometimes be proved more easily by proving its contrapositive instead.



8.1.3 Mathematical Proof

A mathematical proof is a logical sequence of steps, each justified by a known fact, definition, or previously proven result, that establishes a statement is true beyond doubt. Proofs are the mathematical equivalent of the evidence a court or a warranty claim requires — a claim is only accepted once solid justification is presented, not merely asserted.

8.1.4 Theorem, Conjecture, and Axiom

A theorem is a mathematical statement that has been proved true using logical reasoning from previously known facts — for example, that the interior angles of a quadrilateral sum to 360° . A conjecture is a statement believed to be true based on observation but not yet proved, such as the still-unproven Goldbach Conjecture that every even integer greater than 2 is the sum of two primes; if a conjecture is later proved, it becomes a theorem.

An axiom (or postulate, when specifically in geometry) is a statement accepted as true without requiring proof, serving as a foundational starting point for building further mathematical arguments — for example, that a straight line can be drawn between any two points.

8.1.5 Deductive (Algebraic) Proof

A deductive algebraic proof shows that a given equation's left-hand side (LHS) equals its right-hand side (RHS) through a sequence of algebraically valid steps, each one justified explicitly by a law such as the distributive property, commutative law, multiplicative identity, or multiplicative inverse. This is the standard technique for proving algebraic identities like $(x+1)^2 + 7 = x^2 + 2x + 8$.

Important Definitions

Statement

A sentence or mathematical expression that is either true or false, but not both.

Negation ($\sim p$)

The logical operator that reverses the truth value of a statement p ; true when p is false, and false when p is true.



Conjunction ($p \wedge q$)

A compound statement 'p and q' that is true only when both p and q are true.

Disjunction ($p \vee q$)

A compound statement 'p or q' that is true when at least one of p or q is true, and false only when both are false.

Conditional ($p \rightarrow q$)

An 'if p then q' statement, false only when p (the antecedent) is true and q (the consequent) is false.

Biconditional ($p \leftrightarrow q$)

A 'p if and only if q' statement, true exactly when p and q have the same truth value.

Converse, Inverse, Contrapositive

For $p \rightarrow q$: the converse is $q \rightarrow p$, the inverse is $\sim p \rightarrow \sim q$, and the contrapositive is $\sim q \rightarrow \sim p$.

Axiom (Postulate)

A statement accepted as true without requiring proof, used as a starting point for further reasoning; called a postulate specifically in geometry.

Theorem

A mathematical statement that has been proved true using logical reasoning from previously known facts or axioms.

Conjecture

A mathematical statement believed to be true based on observation but not yet proved; becomes a theorem if proved.

Key Facts and Relations

Topic	Key Fact / Relation
$\sim p$ is true iff p is false	Definition of negation.
$p \wedge q$ is true iff both p and q are true	Definition of conjunction.
	Definition of disjunction.



Topic	Key Fact / Relation
$p \vee q$ is true iff at least one of p , q is true	
$p \rightarrow q$ is false iff p is true and q is false	Definition of the conditional.
$p \leftrightarrow q$ is true iff p and q share the same truth value	Definition of the biconditional.
Converse of $p \rightarrow q$ is $q \rightarrow p$	Formed by swapping the antecedent and consequent.
Inverse of $p \rightarrow q$ is $\sim p \rightarrow \sim q$	Formed by negating both the antecedent and consequent.
Contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$	Formed by negating and swapping both parts; always equivalent to $p \rightarrow q$.

Diagrams

The Five Logical Operators: A summary panel listing negation, conjunction, disjunction, conditional, and biconditional with their symbols, readings, and truth rules

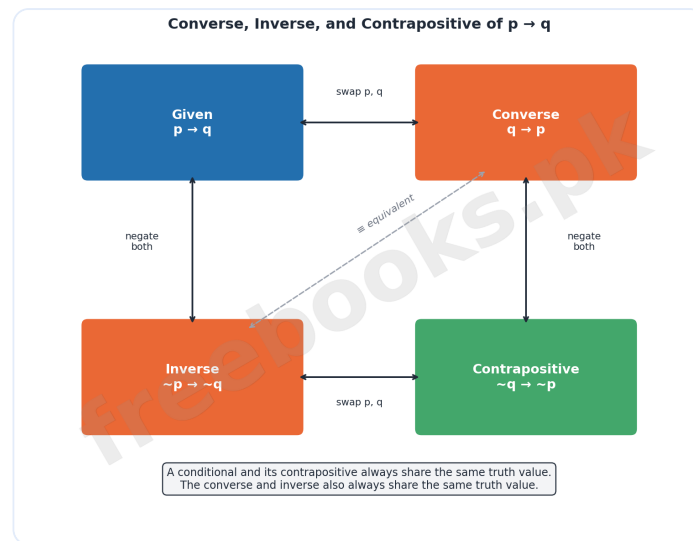


The Five Logical Operators

Converse, Inverse, and Contrapositive of a Conditional: A diagram of four related conditional statements connected by arrows showing how each is

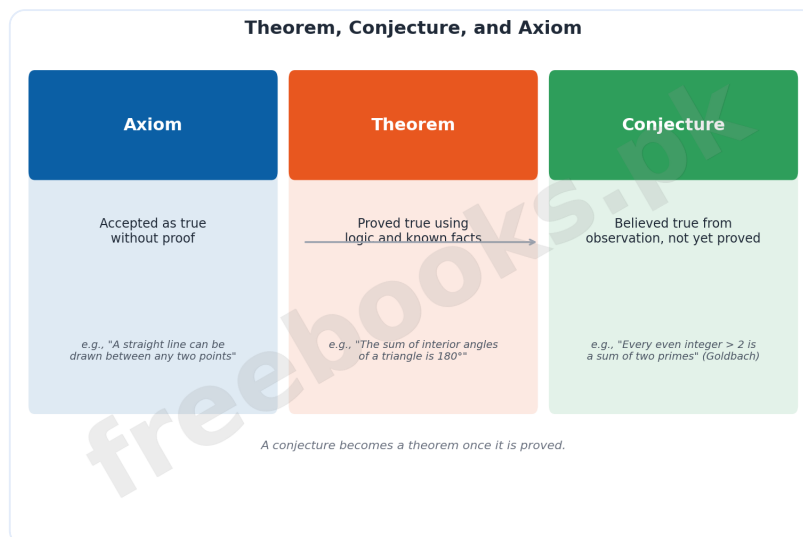


derived, with the equivalence between a conditional and its contrapositive highlighted



Converse, Inverse, and Contrapositive of a Conditional

Theorem, Conjecture, and Axiom: A three-panel comparison of axioms, theorems, and conjectures with definitions and examples, showing how a conjecture becomes a theorem once proved



Theorem, Conjecture, and Axiom



Solved Examples

Example 1: Determining the Truth Value of Conjunctions

Problem: Determine whether each conjunction is true or false: (i) Lahore is the capital of Punjab and Quetta is the capital of Balochistan. (ii) $4 < 5 \wedge 8 < 10$. (iii) $2+2=3 \wedge 6+6=10$.

1. Recall that a conjunction $p \wedge q$ is true only when both p and q are individually true.
2. For (i): both statements are factually true, so the conjunction is true.
3. For (ii): both $4 < 5$ and $8 < 10$ are true, so the conjunction is true.
4. For (iii): both $2+2=3$ and $6+6=10$ are false statements, so the conjunction is false.

Example 2: Constructing a Truth Table for a Compound Statement

Problem: Construct the truth table of $[(p \rightarrow q) \wedge p] \rightarrow q$.

1. List all four combinations of truth values for p and q : TT, TF, FT, FF.
2. Compute $p \rightarrow q$ for each row using its definition (false only when p is T and q is F).
3. Compute $(p \rightarrow q) \wedge p$ for each row by taking the conjunction of the previous column with p .
4. Compute the final column $[(p \rightarrow q) \wedge p] \rightarrow q$; the result is true in all four rows, showing this compound statement is a logical law (tautology).

Example 3: Finding the Converse, Inverse, and Contrapositive

Problem: Given the conditional $p \rightarrow q$, find its converse, inverse, and contrapositive, and state which pairs are logically equivalent.

1. Form the converse by swapping the statements: $q \rightarrow p$.
2. Form the inverse by negating both statements: $\sim p \rightarrow \sim q$.



3. Form the contrapositive by negating and swapping: $\sim q \rightarrow \sim p$.
4. Recall from the truth table that a conditional and its contrapositive always share the same truth value, and the converse and inverse always share the same truth value with each other.

Example 4: Proving a Statement About Odd Integers

Problem: Prove that if x is an odd integer, then x^2 is also an odd integer.

1. Express x using the definition of an odd integer: $x = 2k + 1$ for some integer k .
2. Square both sides: $x^2 = (2k+1)^2 = 4k^2 + 4k + 1$.
3. Factor to isolate a multiple of 2: $x^2 = 2(2k^2+2k) + 1$.
4. Let $m = 2k^2+2k$, an integer, so $x^2 = 2m + 1$, which matches the definition of an odd integer — hence x^2 is odd.

Example 5: Proving a Set Identity Using Logical Steps

Problem: Prove that for any two non-empty sets A and B , $(A \cup B)' = A' \cap B'$.

1. Let $x \in (A \cup B)'$; by definition, $x \notin (A \cup B)$, meaning $x \notin A$ and $x \notin B$.
2. This means $x \in A'$ and $x \in B'$, so $x \in A' \cap B'$, showing $(A \cup B)' \subseteq A' \cap B'$.
3. Conversely, let $y \in A' \cap B'$; then $y \notin A$ and $y \notin B$, so $y \notin (A \cup B)$, meaning $y \in (A \cup B)'$.
4. This shows $A' \cap B' \subseteq (A \cup B)'$; combining both inclusions gives $(A \cup B)' = A' \cap B'$.



Example 6: Classifying a Statement as Axiom, Theorem, or Conjecture

Problem: Classify each statement: (i) There is exactly one straight line through any two points. (ii) Every even number greater than 2 can be written as the sum of two primes. (iii) The sum of the angles in a triangle is 180° .

1. For (i): this is accepted as a basic geometric fact without proof, so it is an axiom (postulate).
2. For (ii): this is the Goldbach Conjecture — believed true from extensive observation but never proved, so it is a conjecture.
3. For (iii): this has been logically proved from more basic geometric facts, so it is a theorem.

Example 7: Formulating a Deductive Algebraic Proof

Problem: Prove that $(x + 1)^2 + 7 = x^2 + 2x + 8$, justifying each step.

1. Start from the left-hand side: $(x+1)^2 + 7 = (x+1)(x+1) + 7$.
2. Apply the distributive law: $x(x+1) + 1(x+1) + 7 = x^2 + x + x + 1 + 7$.
3. Combine like terms using the commutative law: $x^2 + 2x + 8$.
4. Since this matches the right-hand side exactly, $LHS = RHS$, so the identity is proved.

Example 8: Proving a Rational Expression Identity

Problem: Prove that $(45x + 15)/15 = 3x + 1$, justifying each step.

1. Rewrite the left-hand side using division as multiplication: $(1/15) \times (45x + 15)$.
2. Factor 15 out of the bracket: $(1/15) \times 15(3x + 1)$, using the distributive law in reverse.
3. Apply the associative law: $[(1/15) \times 15] \times (3x + 1) = 1 \times (3x + 1)$, using the multiplicative inverse.



4. Apply the multiplicative identity: $1 \times (3x+1) = 3x + 1$, matching the right-hand side, so the identity is proved.

Short Questions & Answers

What makes a sentence a mathematical statement?

It must be either true or false, but not both — this true-or-false property is what distinguishes a statement from other kinds of sentences.

What is the difference between inductive and deductive reasoning?

Inductive reasoning draws general conclusions from repeated observations, while deductive reasoning draws conclusions from premises already accepted as true, guaranteeing the conclusion follows logically.

When is a conditional $p \rightarrow q$ considered false?

Only when the antecedent p is true and the consequent q is false; in every other case it is considered true.

What is the contrapositive of $p \rightarrow q$, and why is it useful?

The contrapositive is $\sim q \rightarrow \sim p$; it is useful because it is always logically equivalent to the original conditional, so a theorem can sometimes be proved more easily by proving its contrapositive instead.

What is the key difference between a theorem and a conjecture?

A theorem has been proved true using logical reasoning from known facts, while a conjecture is only believed true from observation and has not yet been proved.

Why are axioms accepted without proof?

Axioms are basic, self-evident facts that serve as the starting point for building further mathematical arguments; there is nothing more fundamental to prove them from.

What is a deductive algebraic proof used for?

It is used to show that the left-hand side of an equation equals the right-hand side by justifying each algebraic step with a known law, such as the distributive or commutative law.



Long Questions & Answers

Explain the five logical operators and how their truth tables are constructed.

What is negation and how is its truth table built?

Negation $\sim p$ reverses the truth value of p : when p is true, $\sim p$ is false, and when p is false, $\sim p$ is true — a simple two-row truth table.

What are conjunction and disjunction, and how do their truth values differ?

Conjunction $p \wedge q$ requires both p and q to be true to be true overall, while disjunction $p \vee q$ only requires at least one of them to be true — disjunction is false only when both are false.

What is the conditional and why is its truth table often confusing?

The conditional $p \rightarrow q$ is false only when p is true and q is false; in the other three combinations (including when p is false) it is considered true, which can seem counterintuitive but reflects that a false hypothesis makes no claim that can be contradicted.

What is the biconditional, and when is it true?

The biconditional $p \leftrightarrow q$, meaning $p \rightarrow q$ and $q \rightarrow p$ together, is true exactly when p and q share the same truth value — both true, or both false.

Describe how theorems, conjectures, and axioms differ, and how a deductive proof is constructed to establish a theorem.

How do axioms, theorems, and conjectures differ in their basis?

Axioms are accepted as true without proof as foundational starting points; theorems are proved true through logical reasoning from axioms and known facts; conjectures are only believed true from observation and evidence, without proof.



What happens to a conjecture once it is proved?

A proved conjecture becomes a theorem — the Fermat's Last Theorem case shows this transition can take centuries, having remained an unproven conjecture for over 350 years before Andrew Wiles's proof in 1993.

What is the general structure of a deductive algebraic proof?

The proof starts from one side of the equation (typically the LHS), applies a sequence of algebraically valid transformations, each explicitly justified by a named law or property, until it matches the other side (RHS) exactly.

How does justifying each step matter in a deductive proof?

Without explicit justification (e.g., citing the distributive law or multiplicative identity), a proof is just an unverified sequence of algebra; naming the law used at each step is what makes the proof logically rigorous rather than merely suggestive.

Multiple Choice Questions (MCQs)

Which of the following is often associated with inductive reasoning?

- A. based on repeated experiments
- B. if and only if statements
- C. statement proven by a theorem
- D. based on general principles

Answer: A. based on repeated experiments

Explanation: Inductive reasoning draws general conclusions from repeated observations or experiments.

Which sentence best describes deductive reasoning?

- A. general conclusions from limited observations
- B. based on repeated experiments
- C. based on units of accurate information
- D. draw conclusions from well-known facts

Answer: D. draw conclusions from well-known facts

Explanation: Deductive reasoning draws conclusions from premises already accepted or known to be true.



Which of the following statements is true?

- A. The set of integers is finite
- B. The sum of interior angles of any quadrilateral is always 180°
- C. $22/7 \notin \mathbb{Q}'$
- D. All isosceles triangles are equilateral triangles

Answer: C. $22/7 \notin \mathbb{Q}'$

Explanation: $22/7$ is a rational number, so it is correctly stated as not belonging to the set of irrational numbers $\mathbb{Q}'$.

The conjunction of two statements p and q is true when:

- A. both p and q are false
- B. both p and q are true
- C. only q is true
- D. only p is true

Answer: B. both p and q are true

Explanation: A conjunction $p \wedge q$ requires both statements to be true simultaneously.

A conditional is regarded as false only when:

- A. antecedent is true and consequent is false
- B. consequent is true and antecedent is false
- C. antecedent is true only
- D. consequent is false only

Answer: A. antecedent is true and consequent is false

Explanation: This is the sole case in which a conditional $p \rightarrow q$ is considered false.

The contrapositive of $q \rightarrow p$ is:

- A. $q \rightarrow \sim p$
- B. $\sim q \rightarrow p$
- C. $\sim p \rightarrow \sim q$
- D. $\sim p \rightarrow \sim q$ reversed correctly as $\sim p \rightarrow \sim q$

Answer: C. $\sim p \rightarrow \sim q$

Explanation: The contrapositive of $q \rightarrow p$ is formed by negating and swapping both parts, giving $\sim p \rightarrow \sim q$.

The statement 'Every even integer greater than 2 is a sum of two prime numbers' is a:



- A. theorem
- B. conjecture
- C. axiom
- D. postulate

Answer: B. conjecture

Explanation: This is the Goldbach Conjecture — widely believed true but never proved, making it a conjecture.

The statement 'A straight line can be drawn between any two points' is a(n):

- A. theorem
- B. conjecture
- C. axiom
- D. logic

Answer: C. axiom

Explanation: This is one of Euclid's foundational axioms, accepted without proof.

The statement 'The sum of the interior angles of a triangle is 180° ' is a:

- A. converse
- B. theorem
- C. axiom
- D. conditional

Answer: B. theorem

Explanation: This has been logically proved from more basic geometric facts, making it a theorem.

Which best represents the negation of 'The stove is burning'?

- A. The stove is not burning
- B. The stove is dim
- C. The stove is turned to low heat
- D. It is both burning and not burning

Answer: A. The stove is not burning

Explanation: Negation simply reverses the truth value of the original statement without adding new claims.



Quick Revision Summary

- A statement must be either true or false, never both — this is the foundation of all logical reasoning in mathematics.
- Negation, conjunction, disjunction, conditional, and biconditional each have a precise truth table that defines exactly when the compound statement is true.
- A conditional $p \rightarrow q$ is false only when p is true and q is false — this single exception trips up many students.
- A conditional and its contrapositive are always logically equivalent; the converse and inverse are also always equivalent to each other.
- Axioms are accepted without proof, theorems are proved true, and conjectures are believed true but unproven — a conjecture becomes a theorem once proved.
- A deductive algebraic proof must justify every single step with a named law (distributive, commutative, associative, multiplicative identity/inverse) — an unjustified step is not a valid proof.

Exam Tips

- When building a truth table, always list the T/F combinations for p and q in the same consistent order (TT, TF, FT, FF) to avoid errors.
- Remember the conditional's one false case by heart: true antecedent, false consequent — everything else defaults to true.
- To prove a theorem indirectly, try proving its contrapositive instead; since they're equivalent, this sometimes makes the argument far simpler.
- Always define odd/even integers algebraically ($x = 2k+1$ or $x = 2k$) before starting a proof involving them — this is the standard entry point.
- In a deductive algebraic proof, write the specific law used in parentheses next to every step — a proof without justification is incomplete.
- Distinguish carefully between an axiom (no proof needed) and a conjecture (no proof exists yet) — the difference is whether proof is required at all, not just whether one currently exists.



Unit 9: Similar Figures

Two figures are similar when they share the same shape but not necessarily the same size — one is simply a scaled copy of the other. This concept, first formalized by Euclid, is one of the most practically useful ideas in geometry: it underlies map scales, architectural models, photographic enlargements, and countless real-world scaling problems. This unit develops the criteria for identifying similar triangles, quadrilaterals, and solids, then builds the two central formulas of the unit — that the ratio of areas of similar figures equals the square of the ratio of corresponding lengths, and the ratio of volumes of similar solids equals the cube of that same ratio.

The unit closes by applying these ideas to the geometry of regular polygons — their interior angles, exterior angles, diagonals, and symmetry — and to genuinely practical problems: tiling a floor, painting a wall, carpeting a room, and understanding which regular polygons can tessellate a plane without gaps. Throughout, the unifying idea is the scale factor k : lengths scale by k , areas by k^2 , and volumes by k^3 .

Learning Objectives

- Identify similarity of polygons using the AA, SAS, and SSS similarity criteria for triangles.
- Determine whether quadrilaterals and other polygons are similar by checking corresponding angles and side ratios.
- Apply the relationship between the areas of similar figures (ratio of areas = square of ratio of corresponding lengths).
- Apply the relationship between the volumes of similar solids (ratio of volumes = cube of ratio of corresponding lengths).
- Use the mass-volume proportionality of similar solids to solve real-world scaling problems.
- Calculate the interior angle, exterior angle, and number of diagonals of a regular polygon.



- Understand the geometric properties of triangles and parallelograms, including angle sum and exterior angle relationships.
- Determine which regular polygons can tessellate a plane and solve real-life problems involving tiling, painting, carpeting, and fencing.

Key Concepts

9.1 Similarity of Polygons and Triangles

Two polygons are similar if their corresponding angles are equal and their corresponding sides are proportional — one is a scaled version of the other, denoted with the symbol $\sim$. Three criteria identify similar triangles: AA (two pairs of corresponding angles equal, forcing the third pair equal too), SAS (two corresponding sides proportional with their included angle equal), and SSS (all three pairs of corresponding sides proportional).

When one pair of corresponding sides in two triangles is parallel, vertical and alternate angle relationships often show all three angle pairs are equal, proving the triangles similar — a technique used constantly in problems involving intersecting or nested triangles.

9.1.2 Similarity of Quadrilaterals

For quadrilaterals (and other polygons), similarity requires checking both that all corresponding angles are congruent and that all corresponding sides share the same ratio. Missing angles can be found using the fact that a quadrilateral's interior angles sum to 360° , which is often the first step before comparing angle measures.

9.2 Area of Similar Figures

For similar figures with any pair of corresponding lengths ℓ_1 and ℓ_2 , the ratio of their areas satisfies $A_1/A_2 = (\ell_1/\ell_2)^2$. If the scale factor is $k = \ell_1/\ell_2$, then $A_1/A_2 = k^2$ — area scales with the square of the length ratio because area is a two-dimensional quantity built from two length measurements.



9.3 Volume of Similar Solids

Similarly, for similar solids, the ratio of volumes satisfies $V_1/V_2 = (\ell_1/\ell_2)^3 = k^3$ — volume scales with the cube of the length ratio, since volume is a three-dimensional quantity. Because mass is proportional to volume for objects of the same material, the ratio of masses of two similar solids also equals k^3 : $w_1/w_2 = (\ell_1/\ell_2)^3$.

9.4.1 - 9.4.3 Geometrical Properties of Polygons, Triangles, and Parallelograms

For a regular n -sided polygon: the sum of interior angles is $(n-2) \times 180^\circ$, each interior angle measures $[(n-2) \times 180^\circ]/n$, each exterior angle measures $360^\circ/n$, and the total number of diagonals is $n(n-3)/2$. Interior and exterior angles at any vertex are always supplementary (sum to 180°).

In any triangle, the interior angles sum to 180° , and an exterior angle equals the sum of the two non-adjacent (opposite) interior angles. In a parallelogram, opposite sides and opposite angles are equal, adjacent angles are supplementary, and the diagonals bisect each other (but are not necessarily equal, unless it is also a rectangle).

9.4.4 Tessellations and Real-World Applications

A tessellation is a pattern of shapes fitting together with no gaps or overlaps, covering a plane. A regular polygon can tessellate the plane on its own only if its interior angle divides evenly into 360° — this is true for exactly three regular polygons: equilateral triangles (60° , 6 meet at a vertex), squares (90° , 4 meet), and regular hexagons (120° , 3 meet). Regular pentagons and most other regular polygons cannot tessellate alone because their interior angles do not divide evenly into 360° .

These geometric properties directly solve real-world problems: computing the number of tiles or paint needed for a floor or wall, the carpet rolls needed for a room, or the fencing needed around a polygonal plot — all using basic area formulas alongside the similarity and polygon relationships developed in this unit.



Important Definitions

Similar Figures

Figures that have the same shape but not necessarily the same size — one is a scaled copy of the other, with equal corresponding angles and proportional corresponding sides.

Scale Factor (k)

The constant ratio between corresponding lengths of two similar figures, $k = \ell_1 / \ell_2$.

AA, SAS, SSS Similarity

Three sufficient criteria for proving two triangles are similar: two equal angles (AA), two proportional sides with an equal included angle (SAS), or three proportional sides (SSS).

Regular Polygon

A polygon with all sides equal in length and all interior angles equal in measure.

Interior Angle

An angle formed inside a polygon between two adjacent sides; for a regular n -gon, each measures $[(n-2) \times 180^\circ] / n$.

Exterior Angle

The angle formed between one side of a polygon and the extension of an adjacent side; for a regular n -gon, each measures $360^\circ / n$.

Diagonal

A line segment connecting two non-adjacent vertices of a polygon; an n -sided polygon has $n(n-3)/2$ diagonals.

Tessellation

A pattern of one or more shapes that fit together perfectly, without gaps or overlaps, to cover a plane.

Similar Solids

Three-dimensional solids with the same shape but possibly different sizes, having proportional corresponding lengths.

Parallelogram



A quadrilateral with both pairs of opposite sides parallel and equal, opposite angles equal, and diagonals that bisect each other.

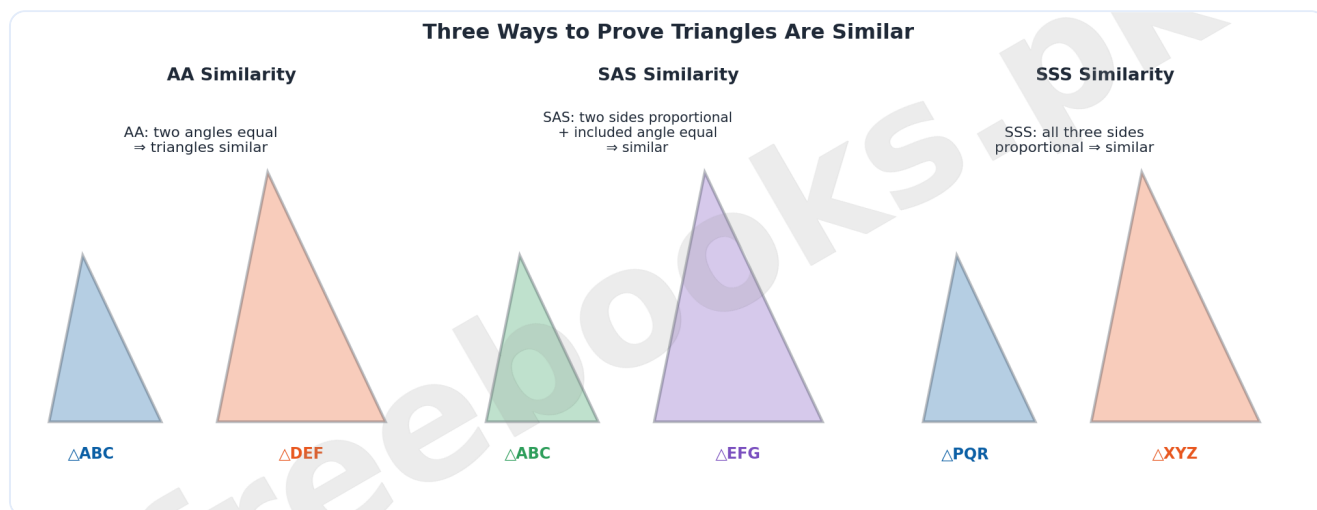
Key Facts and Relations

Topic	Key Fact / Relation
$A_1/A_2 = (\ell_1/\ell_2)^2 = k^2$	Ratio of areas of similar figures equals the square of the ratio of corresponding lengths.
$V_1/V_2 = (\ell_1/\ell_2)^3 = k^3$	Ratio of volumes of similar solids equals the cube of the ratio of corresponding lengths.
$w_1/w_2 = (\ell_1/\ell_2)^3$	Ratio of masses of similar solids equals the cube of the ratio of corresponding lengths (mass proportional to volume).
Sum of interior angles = $(n-2) \times 180^\circ$	Sum of the interior angles of an n-sided polygon.
Interior angle = $[(n-2) \times 180^\circ] / n$	Measure of each interior angle of a regular n-sided polygon.
Exterior angle = $360^\circ / n$	Measure of each exterior angle of a regular n-sided polygon.
Number of diagonals = $n(n-3)/2$	Total number of diagonals in a polygon with n sides.
Interior angle + Exterior angle = 180°	Interior and exterior angles at a vertex are always supplementary.

Diagrams

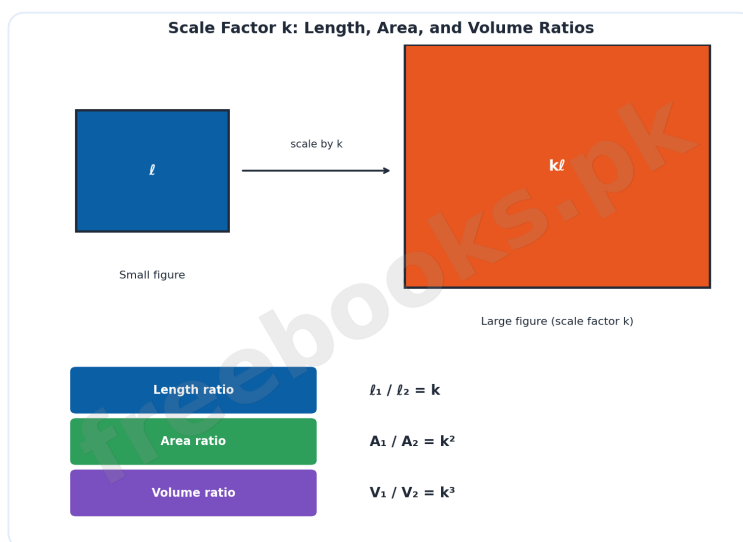
Three Ways to Prove Triangles Are Similar: Three side-by-side triangle pairs illustrating the AA, SAS, and SSS similarity criteria





Three Ways to Prove Triangles Are Similar

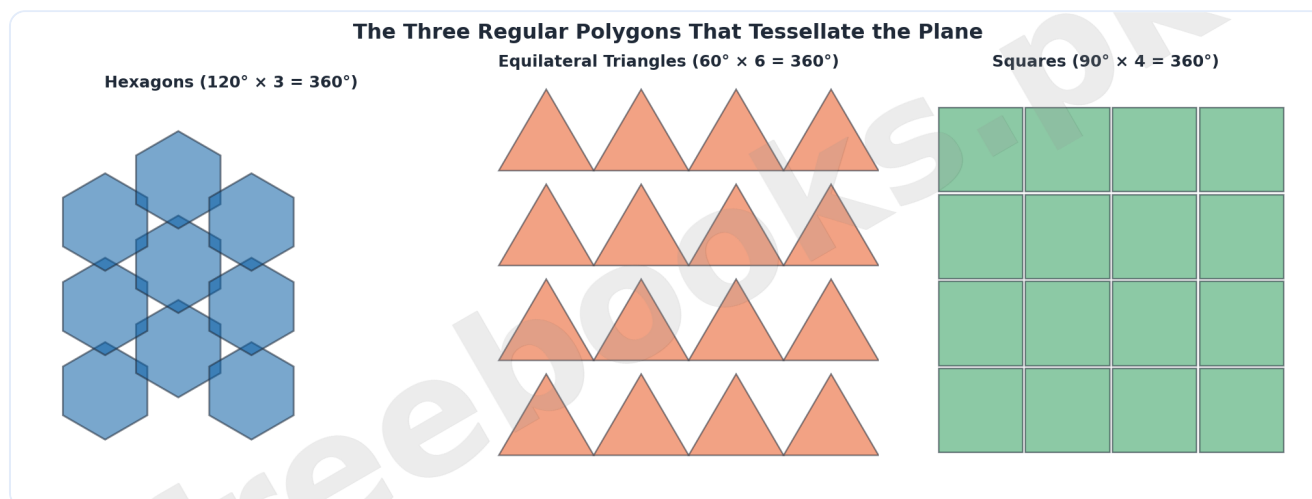
Scale Factor k: Length, Area, and Volume Ratios: A small and large similar figure connected by a scale factor k, with the length, area, and volume ratio formulas listed beside them



Scale Factor k: Length, Area, and Volume Ratios

The Three Regular Polygons That Tessellate the Plane: Three tiling patterns showing hexagons, equilateral triangles, and squares each covering a plane without gaps





The Three Regular Polygons That Tessellate the Plane

Solved Examples

Example 1: Proving Triangles Similar Using the SSS Criterion

Problem: In triangle ABC, the sides are $AB = 6$ cm, $BC = 9$ cm, $CA = 12$ cm. In triangle DEF, the sides are $DE = 10.5$ cm, $EF = 15.75$ cm, $FD = 21$ cm. Prove the triangles are similar.

1. Compute the ratio of each pair of corresponding sides: $AB/DE = 6/10.5 = 4/7$.
2. Compute $BC/EF = 9/15.75 = 4/7$.
3. Compute $CA/FD = 12/21 = 4/7$.
4. Since all three ratios are equal ($4/7$), by the SSS similarity criterion, triangle ABC $\sim$ triangle DEF.

Example 2: Finding an Unknown Area Using the Ratio of Similar Figures

Problem: Two similar triangles have corresponding lengths $\ell_1 = 2.4$ cm and $\ell_2 = 1.5$ cm, with $A_2 = 25$ cm². Find A_1 .

1. Apply the area ratio formula: $A_1/A_2 = (\ell_1/\ell_2)^2$.
2. Substitute the known values: $A_1/25 = (2.4/1.5)^2 = (8/5)^2$.



3. Simplify: $A_1/25 = 64/25$.
4. Solve for A_1 : $A_1 = 64 \text{ cm}^2$.

Example 3: Finding an Unknown Volume Using the Ratio of Similar Solids

Problem: Two similar solids have corresponding lengths $\ell_1 = 5 \text{ cm}$ and $\ell_2 = 7 \text{ cm}$, with $V_2 = 686 \text{ cm}^3$. Find V_1 .

1. Apply the volume ratio formula: $V_1/V_2 = (\ell_1/\ell_2)^3$.
2. Substitute the known values: $V_1/686 = (5/7)^3 = 125/343$.
3. Solve for V_1 : $V_1 = (125/343) \times 686$.
4. Simplify: $V_1 = 250 \text{ cm}^3$.

Example 4: Solving a Similar-Solids Mass Problem

Problem: A sack of rice with height 60 cm has mass 50 kg. Find the mass of a similar sack with height 90 cm.

1. Apply the mass-volume proportionality: $w_1/w_2 = (h_1/h_2)^3$.
2. Substitute the known values: $50/w_2 = (60/90)^3 = (2/3)^3 = 8/27$.
3. Solve for w_2 : $w_2 = (27 \times 50)/8$.
4. Simplify: $w_2 = 168.75 \text{ kg}$.

Example 5: Finding the Interior and Exterior Angle of a Regular Polygon

Problem: Find the measure of each interior angle and each exterior angle of a regular pentagon.

1. Apply the interior angle formula with $n = 5$: $[(5-2) \times 180^\circ]/5 = 540^\circ/5$.
2. Simplify: each interior angle is 108° .
3. Apply the exterior angle formula: $360^\circ/5 = 72^\circ$.



4. Verify supplementarity: $108^\circ + 72^\circ = 180^\circ$, confirming the relationship holds.

Example 6: Determining Whether Two Regular Polygons Can Tessellate Together

Problem: A tessellation combines regular pentagons and regular decagons. Determine whether they can tessellate together at a shared vertex.

1. Find the interior angle of a regular decagon: $[(10-2) \times 180^\circ] / 10 = 1440^\circ / 10 = 144^\circ$.
2. Recall the interior angle of a regular pentagon: 108° .
3. Sum the two angles at the shared vertex: $144^\circ + 108^\circ = 252^\circ$.
4. Since $252^\circ \neq 360^\circ$, the two shapes cannot tessellate together without leaving gaps or overlaps.

Example 7: Using Similar Triangles to Solve a Real-World Shadow Problem

Problem: A man 1.8 m tall casts a shadow 0.76 m long. At the same time, a telephone pole casts a shadow 3 m long. Find the height of the pole.

1. Recognize that the man and his shadow form a triangle similar to the pole and its shadow, since sunlight rays are parallel.
2. Set up the proportion: $(\text{man's height}) / (\text{man's shadow}) = (\text{pole's height}) / (\text{pole's shadow})$.
3. Substitute known values: $1.8 / 0.76 = \text{pole height} / 3$.
4. Solve for the pole's height: $\text{pole height} = (1.8 / 0.76) \times 3 \approx 7.11 \text{ m}$.



Example 8: Real-World Application: Finding the Number of Carpet Rolls Needed

Problem: A parallelogram-shaped room has a base of 10 m and a height of 8 m. Carpet rolls cover 20 m^2 each. How many rolls are needed?

1. Compute the area of the parallelogram-shaped room: $A = \text{base} \times \text{height} = 10 \times 8$.
2. Simplify: $A = 80 \text{ m}^2$.
3. Divide the total area by the coverage per roll: $80/20$.
4. Result: 4 rolls of carpet are needed.

Short Questions & Answers

What condition must be met for two polygons to be similar?

Their corresponding angles must be equal and their corresponding sides must be proportional.

What is the AA similarity criterion for triangles?

If two angles of one triangle are congruent to two corresponding angles of another triangle, the triangles are similar (the third angles must also match automatically).

How does the ratio of areas of similar figures relate to the ratio of their corresponding lengths?

The ratio of areas equals the square of the ratio of corresponding lengths: $A_1/A_2 = (\ell_1/\ell_2)^2$.

How does the ratio of volumes of similar solids relate to the ratio of their corresponding lengths?

The ratio of volumes equals the cube of the ratio of corresponding lengths: $V_1/V_2 = (\ell_1/\ell_2)^3$.

What is the formula for the number of diagonals in an n-sided polygon?

$n(n-3)/2$.



Why do interior and exterior angles of a polygon sum to 180° at each vertex?

Because the interior angle and its adjacent exterior angle together form a straight line, and angles on a straight line always sum to 180° .

Which three regular polygons can tessellate a plane on their own, and why?

Equilateral triangles, squares, and regular hexagons, because their interior angles (60° , 90° , and 120° respectively) divide evenly into 360° , allowing whole numbers of them to meet at each vertex with no gaps.

Long Questions & Answers

Explain the three criteria for proving triangles similar, and how each is applied in a proof.

What is the AA criterion and why is it sufficient?

If two angles of one triangle equal two corresponding angles of another, the triangles are similar; this is sufficient because the third angle is automatically determined (all triangle angles sum to 180°), so all three angle pairs end up equal.

What is the SAS criterion?

If the ratio of two corresponding sides is equal and the angle included between those two sides is also equal, the triangles are similar — matching sides and the angle between them locks the whole triangle's shape.

What is the SSS criterion?

If the ratios of all three pairs of corresponding sides are equal, the triangles are similar, since matching all three side ratios forces the same shape regardless of angles being checked directly.



How are these criteria typically applied together with parallel-line angle facts?

When one pair of sides is parallel, vertical angles and alternate angles from the parallel lines often establish two or three angle equalities directly, allowing the AA (or full angle) criterion to be applied without measuring sides at all.

Describe how area and volume scale with similar figures, and how this applies to real-world scaling problems.

How does area scale with the length ratio of similar figures?

Area is a two-dimensional quantity, so it scales with the square of the length ratio: $A_1/A_2 = (\ell_1/\ell_2)^2 = k^2$, where k is the scale factor.

How does volume scale with the length ratio of similar solids?

Volume is a three-dimensional quantity, so it scales with the cube of the length ratio: $V_1/V_2 = (\ell_1/\ell_2)^3 = k^3$.

How does mass relate to these ratios for similar solids?

Since mass is proportional to volume for objects of the same material, mass also scales with the cube of the length ratio: $w_1/w_2 = (\ell_1/\ell_2)^3$, exactly like volume.

How is this used in a real scaling problem?

Given a known area, volume, or mass at one scale and the ratio of corresponding lengths, the k^2 , or k^3 relationship directly gives the corresponding area, volume, or mass at the other scale — as in scaling a sack of rice's mass from a smaller to a larger similar sack using the cube of the height ratio.

Multiple Choice Questions (MCQs)

If two polygons are similar, then:

- A. their corresponding angles are equal
- B. their areas are equal
- C. their volumes are equal
- D. their corresponding sides are equal



Answer: A. their corresponding angles are equal

Explanation: Similar polygons have equal corresponding angles and proportional (not necessarily equal) corresponding sides.

The ratio of the areas of two similar polygons is:

- A. equal to the ratio of their perimeters
- B. equal to the square of the ratio of their corresponding sides
- C. equal to the cube of the ratio of their corresponding sides
- D. equal to the sum of their corresponding sides

Answer: B. equal to the square of the ratio of their corresponding sides

Explanation: Area ratio equals the square of the length ratio, $A_1/A_2 = (\ell_1/\ell_2)^2$.

If the volumes of two similar solids are 125 cm^3 and 27 cm^3 , the ratio of their corresponding heights is:

- A. 3:5
- B. 5:3
- C. 25:9
- D. 9:25

Answer: B. 5:3

Explanation: Taking the cube root of 125:27 gives 5:3, since $(5/3)^3 = 125/27$.

The exterior angle of a regular pentagon is:

- A. 40°
- B. 45°
- C. 60°
- D. 72°

Answer: D. 72°

Explanation: Exterior angle = $360^\circ/5 = 72^\circ$.

A parallelogram has area 64 cm^2 and a similar parallelogram has area 144 cm^2 . If a side of the smaller parallelogram is 8 cm, the corresponding side of the larger is:

- A. 10 cm
- B. 12 cm
- C. 18 cm
- D. 16 cm

Answer: B. 12 cm



Explanation: Area ratio 64:144 simplifies to 4:9, so length ratio is 2:3; $8 \times (3/2) = 12$ cm.

The total number of diagonals in a polygon with 9 sides is:

- A. 18
- B. 21
- C. 25
- D. 27

Answer: D. 27

Explanation: Using $n(n-3)/2 = 9 \times 6/2 = 27$.

Two spheres are similar with radii in ratio 4:5. If the surface area of the larger sphere is 500π cm², the surface area of the smaller sphere is:

- A. 256π cm²
- B. 320π cm²
- C. 400π cm²
- D. 405π cm²

Answer: B. 320π cm²

Explanation: Area ratio = $(4/5)^2 = 16/25$; smaller area = $500\pi \times 16/25 = 320\pi$ cm².

A regular polygon has an exterior angle of 30°. How many sides does it have?

- A. 8
- B. 10
- C. 12
- D. 15

Answer: C. 12

Explanation: Number of sides = $360^\circ/30^\circ = 12$.

In a regular hexagon, the ratio of the length of the longest diagonal to the side length is:

- A. $\sqrt{3} : 1$
- B. $2 : 1$
- C. $3 : 2$
- D. $2 : 3$

Answer: B. $2 : 1$



Explanation: The longest diagonal of a regular hexagon passes through the center and equals twice the side length, giving ratio 2:1.

A regular polygon has an interior angle of 165° . How many sides does it have?

- A. 15
- B. 16
- C. 20
- D. 24

Answer: D. 24

Explanation: Exterior angle = $180^\circ - 165^\circ = 15^\circ$; number of sides = $360^\circ / 15^\circ = 24$.

Quick Revision Summary

- Two polygons are similar when corresponding angles are equal and corresponding sides are proportional; triangles specifically can be proved similar via AA, SAS, or SSS.
- Area ratio of similar figures = square of the length ratio ($A_1/A_2 = k^2$); volume ratio of similar solids = cube of the length ratio ($V_1/V_2 = k^3$).
- Mass of similar solids scales the same way as volume, since mass is proportional to volume for the same material.
- Interior angle of a regular n -gon = $[(n-2) \times 180^\circ]/n$; exterior angle = $360^\circ/n$; number of diagonals = $n(n-3)/2$.
- Interior and exterior angles at any polygon vertex are always supplementary (sum to 180°).
- Only equilateral triangles, squares, and regular hexagons can tessellate a plane alone, because their interior angles divide evenly into 360° .

Exam Tips

- When proving triangle similarity, look first for parallel lines — they usually hand you two angle equalities immediately via alternate or vertical angles.
- Always double-check which ratio (l_1/l_2 or l_2/l_1) matches which area/volume in the formula before substituting — flipping it is a common error.



- Remember: lengths scale by k , areas by k^2 , volumes by k^3 — write this out before starting any similar-figures word problem.
- For polygon-angle problems, always compute n from the given angle first (via $360^\circ/\text{exterior angle}$) before answering further sub-questions.
- To test if two regular polygons can tessellate together, always check whether their interior angles sum to exactly 360° at the shared vertex.
- In real-world area/tiling problems, compute the area of the room or wall first, then divide by the coverage of a single tile, roll, or gallon — do the division last.



Unit 10: Graphs of Functions

A graph turns an abstract equation into a visual shape, making it far easier to understand how one quantity changes in response to another. This unit builds a visual toolkit for the major function families encountered in school mathematics: linear functions, whose graphs are straight lines; quadratic and cubic functions, whose graphs are curved parabolas and S-shaped curves respectively; reciprocal functions, whose graphs split into separate branches approaching invisible boundary lines called asymptotes; and exponential functions, whose graphs capture the runaway growth or steady decay seen throughout nature and finance.

Beyond simply plotting points, the unit teaches how to read meaning out of a graph — estimating a value at a point that wasn't directly calculated, finding the steepness (gradient) of a curve at a specific point by drawing its tangent line, and recognizing exponential growth or decay from a graph's shape alone. The unit closes with genuinely practical applications: sketching a salary function to see how pay grows with experience, and graphing cost and revenue functions together to find a business's break-even point — the exact production level where profit turns from negative to positive.

Learning Objectives

- Recall and sketch graphs of linear functions of the form $y = mx + c$.
- Plot and interpret graphs of quadratic functions $y = ax^2 + bx + c$, recognizing the parabola shape and its direction of opening.
- Plot and interpret graphs of cubic functions $y = ax^3 + bx^2 + cx + d$ and describe their general S-shaped behavior.
- Plot and interpret graphs of reciprocal functions $y = a/x$, including their asymptotic behavior.
- Graph $y = ax^n$ for positive integer, negative integer, and rational values of n (for $x > 0$).
- Graph exponential functions $y = ka^x$ ($a > 1$) and distinguish exponential growth from decay.



- Determine the gradient of a curve at a point by drawing a tangent line and computing its slope.
- Apply graph sketching and interpretation to real-life problems such as salary growth, tax payment, and cost-profit break-even analysis.

Key Concepts

10.1 Functions and Their Graphs

A function $y = f(x)$ expresses a dependent variable y in terms of an independent variable x — f can be represented as an equation, a graph, a table of values, or a verbal description. Graphing a function means plotting the set of all points $(x, f(x))$ that satisfy the relationship, giving a visual picture of how output changes as input changes.

10.1.1 - 10.1.2 Linear and Quadratic Functions

A linear function $f(x) = mx + c$ graphs as a straight line, with m controlling the steepness (slope) and c giving the y -intercept. Its graph is sketched most easily by finding the x - and y -intercepts (setting $y = 0$ and $x = 0$ respectively) and joining them with a straight line.

A quadratic function $y = ax^2 + bx + c$ ($a \neq 0$) always graphs as a parabola: it opens upward if $a > 0$ and downward if $a < 0$. The parabola's key features — where it crosses the x -axis and y -axis, and its turning point — are found from a table of values or algebraic analysis.

10.1.3 - 10.1.4 Cubic and Reciprocal Functions

A cubic function $y = ax^3 + bx^2 + cx + d$ ($a \neq 0$) graphs as a curve with an S-shaped appearance and at most two turning points; its exact shape varies more than linear or quadratic graphs depending on the coefficients.

A reciprocal function $y = a/x$ ($x \neq 0$) produces two separate curve branches — commonly one in each of two opposite quadrants — that approach the x -axis and y -axis (its asymptotes) but never touch them, since the function is undefined at $x = 0$.



10.1.5 - 10.1.6 Exponential Functions and $y = ax^n$

An exponential function $y = ka^x$ ($a > 1$) grows increasingly steeply as x increases — this shape represents growth, common in population models, compound interest, and epidemic spread. The constant $e \approx 2.72$ gives the especially important natural exponential function $y = e^x$.

The general power function $y = ax^n$ behaves very differently depending on n : for positive integer n (like $n=3$) the graph passes through the origin and rises; for negative integer n (like $n=-1$) the graph splits into branches approaching the axes as asymptotes; for rational n (like $n=1/5$) the graph has a smoother, flatter growth curve for $x > 0$.

10.2 Exponential Growth/Decay and Gradients of Curves

Exponential growth graphs (e.g., population, compound interest) start slowly and accelerate sharply upward over time; exponential decay graphs (e.g., cooling, depreciation) start high and fall sharply before leveling off. Reading these graphs allows estimation of a quantity at any given time without recomputing the full equation.

The gradient of a curve at a specific point equals the gradient of the tangent line drawn at that point — a straight line touching the curve at exactly one point without crossing it. Once two points on the tangent line are identified, the gradient is computed the same way as for any straight line: $\Delta y / \Delta x = (y_2 - y_1) / (x_2 - x_1)$.

10.2.2 Real-Life Applications

Graphing directly supports practical decision-making: sketching salary versus years of experience reveals the rate of pay growth; graphing tax liability against income can reveal bracket thresholds; and graphing a cost function alongside a revenue function reveals the break-even point — the production level where revenue exactly equals cost, found either graphically (where the two lines intersect) or algebraically (by setting $R(x) = C(x)$ and solving for x).



Important Definitions

Function

A relationship $y = f(x)$ in which each input value x produces exactly one output value y .

Linear Function

A function of the form $f(x) = mx + c$, whose graph is a straight line with slope m and y-intercept c .

Quadratic Function

A function of the form $y = ax^2 + bx + c$ ($a \neq 0$), whose graph is a parabola opening upward if $a > 0$ or downward if $a < 0$.

Cubic Function

A function of the form $y = ax^3 + bx^2 + cx + d$ ($a \neq 0$), whose graph is an S-shaped curve with at most two turning points.

Reciprocal Function

A function of the form $y = a/x$ ($x \neq 0$), whose graph consists of two separate branches approaching the axes as asymptotes.

Asymptote

A line that a graph approaches ever more closely but never actually touches or crosses.

Exponential Function

A function of the form $y = ka^x$ ($a > 1$), whose graph rises with increasing steepness, representing growth.

Tangent Line

A straight line that touches a curve at exactly one point without crossing it there.

Gradient of a Curve at a Point

The slope of the tangent line drawn to the curve at that specific point, computed as $\Delta y / \Delta x$.

Break-Even Point

The production or sales level at which total revenue exactly equals total cost, so profit is zero — found by solving $R(x) = C(x)$.



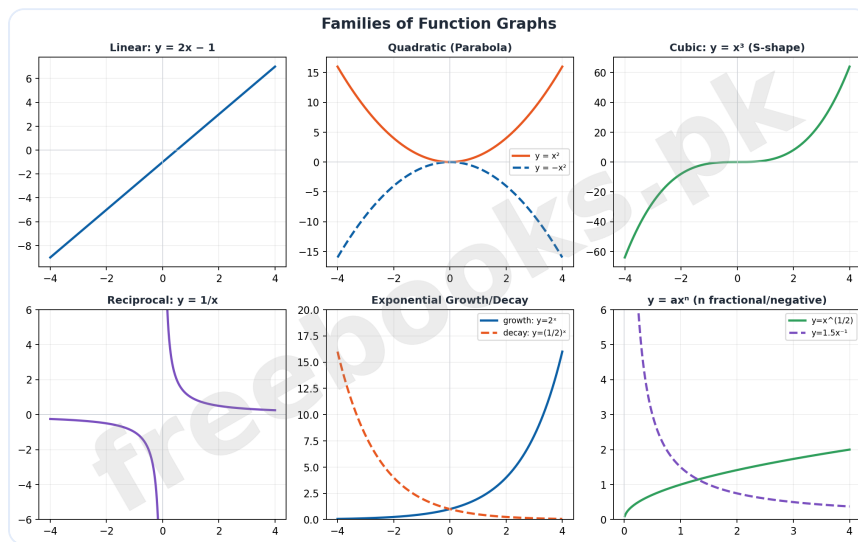
Key Facts and Relations

Topic	Key Fact / Relation
$f(x) = mx + c$	General form of a linear function; m is the slope, c is the y -intercept.
$y = ax^2 + bx + c, a \neq 0$	General form of a quadratic function, graphing as a parabola.
$y = ax^3 + bx^2 + cx + d, a \neq 0$	General form of a cubic function, graphing as an S-shaped curve.
$y = a/x, x \neq 0$	General form of a reciprocal function, with the axes as asymptotes.
$y = ka^x, a > 1$	General form of an exponential growth function.
$y = ax^n$	General power function; behavior depends on whether n is a positive integer, negative integer, or rational number.
Gradient = $\Delta y / \Delta x = (y_2 - y_1) / (x_2 - x_1)$	Formula for the gradient of a line or tangent, using two points on it.
Break-even: $R(x) = C(x)$	Condition defining the break-even point, where revenue equals cost.
$P(x) = R(x) - C(x)$	Profit function, equal to revenue minus cost.

Diagrams

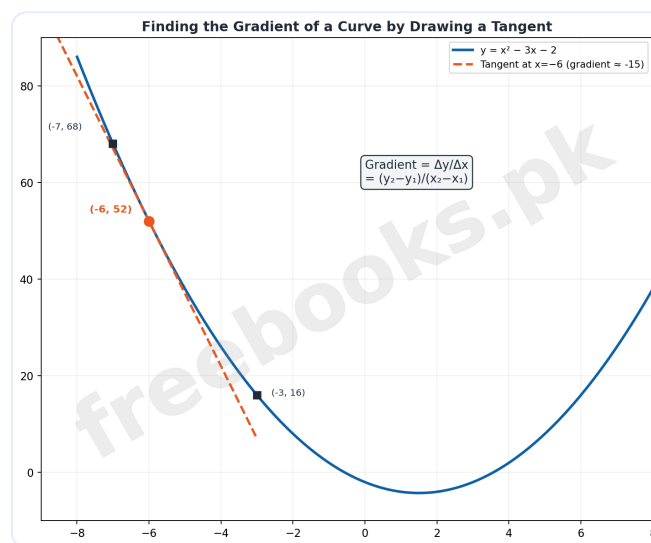
Families of Function Graphs: A six-panel gallery showing the characteristic shapes of linear, quadratic, cubic, reciprocal, exponential, and power (ax^n) function graphs





Families of Function Graphs

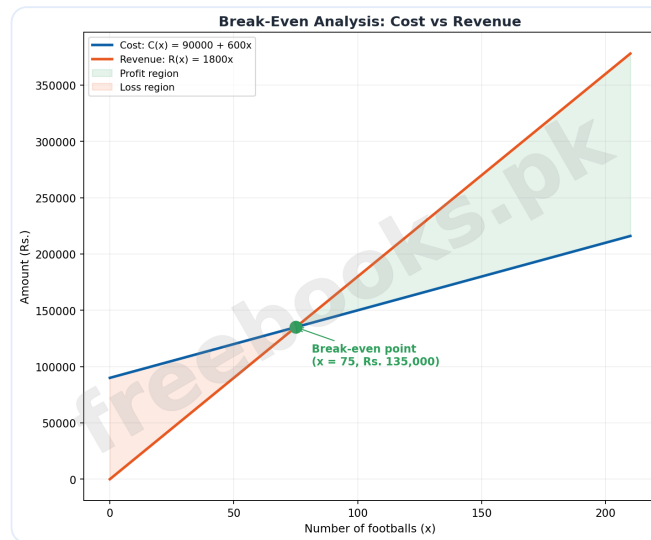
Finding the Gradient of a Curve by Drawing a Tangent: A parabola with a tangent line drawn at a point, showing how two points on the tangent give the gradient via $\Delta y / \Delta x$



Finding the Gradient of a Curve by Drawing a Tangent

Break-Even Analysis: Cost vs Revenue: Cost and revenue lines plotted together, showing their intersection at the break-even point with profit and loss regions shaded





Break-Even Analysis: Cost vs Revenue

Solved Examples

Example 1: Sketching a Linear Function Using Intercepts

Problem: Sketch the graph of $y = 2x - 1$.

1. Find the y-intercept by setting $x = 0$: $y = 2(0) - 1 = -1$, giving the point $(0, -1)$.
2. Find the x-intercept by setting $y = 0$: $0 = 2x - 1$, so $x = 1/2$, giving the point $(1/2, 0)$.
3. Plot both intercepts on the coordinate plane and join them with a straight line.
4. Since the slope (2) is positive, the line rises from left to right.

Example 2: Sketching a Quadratic Function From a Table of Values

Problem: Sketch the graph of $y = 2x^2 - 7x - 9$ for $-3 \leq x \leq 6$.

1. Build a table of x and corresponding y values across the given interval by substituting each x into the equation.
2. Plot each (x, y) pair as a point on the coordinate plane.



3. Connect the points with a smooth curve, since a is positive ($a=2$), confirming the parabola opens upward.
4. Read off where the curve crosses the axes: y -axis at $(0, -9)$, x -axis near $(-1, 0)$ and $(4.5, 0)$.

Example 3: Plotting a Reciprocal Function and Identifying Its Asymptotes

Problem: Sketch the graph of $y = 1/(x - 0.5)$, $x \neq 0.5$.

1. Build a table of values on either side of the undefined point $x = 0.5$, using values close to and far from it.
2. Note that as x approaches 0.5 from either side, y grows without bound — this identifies $x = 0.5$ as a vertical asymptote.
3. Note that as x becomes very large or very negative, y approaches 0 — this identifies $y = 0$ as a horizontal asymptote.
4. Plot the two separate branches of the curve, each approaching but never touching these asymptotes.

Example 4: Modeling Exponential Population Growth From a Graph

Problem: A village's population follows $p = 753e^{(0.03t)}$. Graph the population from $t=0$ to $t=30$ and estimate the population in 2020 ($t=10$) and 2030 ($t=20$).

1. Build a table of (t, p) values by substituting $t = 0, 5, 10, \dots, 30$ into the equation.
2. Plot the points and connect them with a smooth upward curve, characteristic of exponential growth.
3. Read the estimated value directly off the graph at $t = 10$: approximately 1016 persons (year 2020).



4. Read the estimated value directly off the graph at $t = 20$: approximately 1372 persons (year 2030).

Example 5: Finding the Gradient of a Curve by Drawing a Tangent

Problem: Sketch $y = x^2 - 3x - 2$ for $-8 \leq x \leq 8$, draw a tangent at $x = -6$, and determine its gradient.

1. Build a table of values and plot the curve across the given interval.
2. Draw a straight line that just touches the curve at $x = -6$ without crossing it — the tangent line.
3. Choose two clear points on the tangent line, such as $(-3, 10)$ and $(-7, 70)$.
4. Compute the gradient: $(70 - 10)/(-7 - (-3)) = 60/(-4) = -15$, indicating the curve is decreasing as x increases at this point.

Example 6: Real-World Application: Interpreting a Linear Salary Graph

Problem: Majid's salary is $S(x) = 25000 + 1500x$, where x is years worked. Sketch and interpret the graph for $0 \leq x \leq 10$.

1. Build a table of values by substituting $x = 0, 2, 4, 6, 8, 10$ into the salary formula.
2. Plot the points; since the formula is linear, they lie on a straight line.
3. Observe the y-intercept, 25000, representing the starting salary at $x = 0$.
4. Interpret the slope, 1500, as the fixed salary increase (in rupees) for every additional year worked.



Example 7: Real-World Application: Finding a Break-Even Point Graphically and Algebraically

Problem: A company's cost is $C(x) = 90,000 + 600x$ and revenue is $R(x) = 1,800x$ for producing/selling x footballs. Find the break-even point and the profit at $x=150$.

1. Set revenue equal to cost to find the break-even point algebraically: $1800x = 90000 + 600x$.
2. Solve: $1200x = 90000$, so $x = 75$ — the break-even point occurs at 75 footballs.
3. Compute revenue and cost at $x = 150$: $R(150) = \text{Rs. } 270,000$, $C(150) = \text{Rs. } 180,000$.
4. Compute the profit: $P(150) = R(150) - C(150) = \text{Rs. } 270,000 - \text{Rs. } 180,000 = \text{Rs. } 90,000$.

Example 8: Comparing Exponential Growth and Decay Shapes

Problem: Plot $y = 2^x$ and $y = 2^{-x}$ on the same diagram for $-4 \leq x \leq 4$ and describe the difference in their shapes.

1. Build a table of values for $y = 2^x$, noting it grows rapidly as x increases and approaches 0 as x decreases.
2. Build a table of values for $y = 2^{-x} = (1/2)^x$, noting it falls rapidly as x increases and grows large as x decreases.
3. Plot both curves on the same axes, observing they are mirror images of each other across the y -axis.
4. Conclude that $y = 2^x$ represents exponential growth while $y = 2^{-x}$ represents exponential decay.



Short Questions & Answers

What information do the x- and y-intercepts give when sketching a linear function?

They provide two points that lie exactly on the line, which is enough to draw the entire straight-line graph.

How does the sign of a in $y = ax^2 + bx + c$ affect the parabola's shape?

If $a > 0$ the parabola opens upward; if $a < 0$ the parabola opens downward.

What is an asymptote?

A line that a graph approaches increasingly closely but never actually touches or crosses.

What distinguishes an exponential growth graph from an exponential decay graph?

A growth graph rises increasingly steeply as x increases, while a decay graph falls sharply and then levels off as x increases.

How is the gradient of a curve at a specific point found?

By drawing the tangent line to the curve at that point, then computing the gradient of that tangent line using two points on it.

What does the y-intercept of a salary function like $S(x) = 25000 + 1500x$ represent?

The starting salary before any years of service have been counted, i.e., the salary at $x = 0$.

How is the break-even point of a business found algebraically?

By setting the revenue function equal to the cost function, $R(x) = C(x)$, and solving for x .



Long Questions & Answers

Describe the characteristic graph shapes of linear, quadratic, cubic, reciprocal, and exponential functions.

What does a linear function's graph look like?

A straight line, with the slope m controlling its steepness and direction, and the y-intercept c marking where it crosses the y-axis.

What does a quadratic function's graph look like?

A parabola, always symmetric about a vertical line through its turning point, opening upward if the leading coefficient is positive and downward if negative.

What does a cubic function's graph look like?

An S-shaped curve with at most two turning points, showing more varied behavior than linear or quadratic graphs depending on its coefficients.

What do reciprocal and exponential graphs look like?

A reciprocal function's graph splits into two branches approaching the x- and y-axes as asymptotes without touching them; an exponential function's graph rises (or falls, for decay) with continuously changing steepness, never touching the x-axis.

Explain how tangent lines are used to find the gradient of a curve, and how break-even analysis uses two graphed functions together.

Why can't the ordinary two-point slope formula be applied directly to a curve?

A curve's steepness constantly changes from point to point, unlike a straight line, so there is no single overall slope — the gradient must instead be found locally at each specific point of interest.



How does a tangent line solve this problem?

By drawing a straight line that touches the curve at exactly one chosen point without crossing it, the (locally straight) tangent line can be treated as an ordinary line, and its gradient is computed the standard way from any two of its own points.

How is a break-even point identified using two functions?

The cost function $C(x)$ and revenue function $R(x)$ are graphed on the same axes; wherever their curves intersect, revenue exactly equals cost, marking the break-even production level.

How does the graph show profit versus loss regions?

To the left of the break-even point, cost exceeds revenue, indicating a loss region; to the right, revenue exceeds cost, indicating a profit region — visually splitting the graph into two economically meaningful zones.

Multiple Choice Questions (MCQs)

$x = 5$ represents:

- A. x-axis
- B. y-axis
- C. line parallel to x-axis
- D. line parallel to y-axis

Answer: D. line parallel to y-axis

Explanation: $x = 5$ is a vertical line at x-coordinate 5, parallel to the y-axis.

The slope of the line $y = 5x + 3$ is:

- A. 3
- B. -3
- C. 5
- D. -5

Answer: C. 5

Explanation: In $y = mx + c$ form, the coefficient of x , which is 5, is the slope.

The y-intercept of $y = -2x - 1$ is:



- A. -2
- B. 2
- C. -1
- D. 1

Answer: C. -1

Explanation: Setting $x = 0$ gives $y = -1$, the y-intercept.

The graph of $y = x^3$ cuts the x-axis at:

- A. $x = 0$
- B. $x = 1$
- C. $x = -1$
- D. $x = 2$

Answer: A. $x = 0$

Explanation: Setting $y = 0$ gives $x^3 = 0$, so $x = 0$ is the only root.

The graph of $y = 3^x$ represents:

- A. growth
- B. decay
- C. both growth and decay
- D. a straight line

Answer: A. growth

Explanation: Since the base $3 > 1$, the graph rises with increasing x , representing exponential growth.

The graph of $y = -x^2 + 5$ opens:

- A. upward
- B. downward
- C. left
- D. right

Answer: B. downward

Explanation: The coefficient of x^2 is negative (-1), so the parabola opens downward.

The graph of $y = x^2 - 9$ opens:

- A. upward
- B. downward
- C. left
- D. right



Answer: A. upward

Explanation: The coefficient of x^2 is positive (1), so the parabola opens upward.

$y = 5^x$ is a(n) _____ function.

- A. linear
- B. quadratic
- C. cubic
- D. exponential

Answer: D. exponential

Explanation: A function with a constant base raised to a variable power is exponential.

Which of the following is a reciprocal function?

- A. $y = 7^x$
- B. $y = 2/x$
- C. $y = 2x^2$
- D. $y = 5x^3$

Answer: B. $y = 2/x$

Explanation: $y = 2/x$ has the form $y = a/x$, the defining form of a reciprocal function.

$y = -3x^3 + 7$ is a(n) _____ function.

- A. exponential
- B. cubic
- C. linear
- D. reciprocal

Answer: B. cubic

Explanation: The highest power of x is 3, matching the general cubic form ax^3+bx^2+cx+d .

Quick Revision Summary

- A linear function's graph is found quickly using its x - and y -intercepts, or its slope and one known point.
- A quadratic function's graph is a parabola opening upward if $a>0$ and downward if $a<0$; a cubic function's graph is an S-shaped curve with up to two turning points.



- A reciprocal function's graph has two branches approaching asymptotes (lines it never touches) at the axes.
- An exponential function $y = ka^x$ ($a > 1$) grows increasingly steeply; its mirror image $y = ka^{-x}$ decays, flattening out over time.
- The gradient of a curve at a point is found by drawing its tangent line there and computing that line's slope using two of its own points.
- A break-even point is found by setting a business's revenue function equal to its cost function and solving for the production level x .

Exam Tips

- Always build a table of values before attempting to sketch any nonlinear function — a handful of accurate points prevents guessing the curve's shape.
- For quadratics, check the sign of the leading coefficient first — it immediately tells you whether the parabola opens up or down before any plotting.
- For reciprocal functions, plot values very close to the undefined x -value on both sides — this reveals the asymptote's direction and behavior clearly.
- When drawing a tangent by eye, choose two points on the tangent line itself (not on the curve) that are far apart, for a more accurate gradient calculation.
- In break-even problems, always solve $R(x) = C(x)$ algebraically first, then use the graph as a visual confirmation of the same answer.
- Distinguish carefully between exponential growth ($a > 1$) and decay ($0 < a < 1$, or equivalently a negative exponent) — the base or exponent sign is the deciding factor.



Unit 11: Loci and Construction

Geometric construction is the practice of drawing accurate figures using only a ruler and compass, without relying on a protractor for measurements that can be produced more precisely by construction. This unit begins with the four classic cases of triangle construction — three sides (SSS), two sides with the included angle (SAS), one side with two angles (ASA), and two sides with an angle opposite one of them (SSA, the famous ambiguous case that can yield two triangles, one triangle, or none at all).

The unit then introduces the special lines of a triangle — perpendicular bisectors, medians, angle bisectors, and altitudes — and shows that each set of three lines meets at a single special point: the circumcentre, centroid, incentre, and orthocentre respectively. Finally, the unit introduces the idea of a locus: the set of all points obeying one geometric rule. The four basic loci in two dimensions (a circle, a pair of parallel lines, a perpendicular bisector, and an angle bisector) are constructed individually and then combined, since real positioning problems — from placing a fire hydrant to designing a robotic arm's safe zone — are usually solved by finding where two or more loci intersect.

Learning Objectives

- Construct a triangle given its three sides (SSS) using ruler and compass.
- Construct a triangle given two sides and the included angle (SAS).
- Construct a triangle given one side and two angles (ASA).
- Construct a triangle given two sides and an angle opposite one of them (SSA), and determine the number of possible triangles in the ambiguous case.
- Apply the Triangle Inequality Theorem to test whether three given lengths can form a triangle.
- Construct the perpendicular bisectors, medians, angle bisectors, and altitudes of a triangle, and identify their points of concurrency: circumcentre, centroid, incentre, and orthocentre.
- Define a locus and construct the four basic loci in two dimensions: a circle, parallel lines, a perpendicular bisector, and an angle bisector.



- Solve construction problems by finding the intersection of two or more loci, and apply loci to real-life positioning problems.

Key Concepts

11.1 Construction of Triangles

A triangle can be constructed accurately with ruler and compass from three independent pieces of information, covering four standard cases. In the SSS case, all three sides are given: one side is drawn first, then arcs of the other two side-lengths, centred at its two endpoints, are drawn to intersect at the third vertex. In the SAS case, two sides and the angle between them are given: one side is drawn, an angle is constructed at one endpoint, and the second given side is marked off along the new ray.

In the ASA case, one side and two angles are given: the side is drawn first, then angles are constructed at both of its endpoints, with the two new rays intersecting at the third vertex. The SSA case — two sides and an angle opposite one of them — is different: it is called the ambiguous case because, depending on the exact measurements, it can produce two distinct triangles, exactly one triangle (when the given opposite side exactly equals the perpendicular distance from the third vertex, giving a right angle at the point of tangency), or no triangle at all (when the given side is too short to reach the required vertex).

Before attempting any construction, the Triangle Inequality Theorem should be checked: the sum of the lengths of any two sides of a triangle must always be greater than the length of the third side, otherwise the three given lengths cannot form a triangle at all. Triangles are further classified by their sides (scalene: all sides different; isosceles: two sides equal; equilateral: all three sides equal) and by their angles (acute: all angles less than 90° ; right: one angle exactly 90° ; obtuse: one angle greater than 90°).

11.2 Perpendicular Bisectors and Medians of a Triangle

A perpendicular bisector of a triangle's side is a line that crosses that side at its exact midpoint and at a right angle (90°) to it; it is constructed by drawing equal-radius arcs from both endpoints of the side, above and below it, and joining the



two points where the arcs cross. A median is a line segment joining a vertex of the triangle directly to the midpoint of the side opposite that vertex.

Although a triangle has three perpendicular bisectors (one per side) and three medians (one per vertex), each set of three lines always meets at exactly one shared point — a property called concurrency. The three perpendicular bisectors of a triangle are concurrent at the circumcentre, a point equidistant from all three vertices (and therefore the centre of the triangle's circumscribed circle). The three medians are concurrent at the centroid, the triangle's balance point, which always lies inside the triangle regardless of its shape.

11.3 Angle Bisector of a Triangle

An angle bisector of a triangle is a ray that starts at one vertex and splits that vertex's interior angle into two exactly equal halves; it is constructed by drawing an arc from the vertex that cuts both sides of the angle at equal distances, then drawing two further equal-radius arcs from those two points to find a crossing point inside the angle, which is then joined to the vertex.

The three angle bisectors of a triangle — one from each vertex — are always concurrent at a single point called the incentre. The incentre is equidistant from all three sides of the triangle (not the vertices), which makes it the centre of the triangle's inscribed circle (incircle), the largest circle that fits entirely inside the triangle while touching all three sides.

11.4 Altitudes of a Triangle

An altitude of a triangle is a perpendicular line segment drawn from one vertex to the line containing the side opposite that vertex (extending that side if necessary). Each triangle has three altitudes, one from each vertex, and they are constructed using a set square or a standard perpendicular-line compass construction.

The three altitudes of a triangle are always concurrent at a single point called the orthocentre. The orthocentre's position relative to the triangle depends on the triangle's shape: it lies strictly inside an acute triangle, exactly at the right-angle vertex of a right triangle, and outside an obtuse triangle (requiring the altitudes to be extended beyond the triangle's sides to find the meeting point).



11.5 Loci and Construction

A locus (plural: loci) is the set of all points in a plane that satisfy one specific given geometric condition or rule. Rather than plotting isolated points one at a time, a locus is constructed as a continuous curve, line, or region representing every point that obeys the stated rule. Four basic loci recur throughout geometry: the locus of points at a fixed distance from a fixed point is a circle (centred at that point, with the fixed distance as radius); the locus of points at a fixed distance from a fixed straight line is a pair of straight lines running parallel to it, one on each side; the locus of points equidistant from two fixed points is the perpendicular bisector of the segment joining them; and the locus of points equidistant from two intersecting straight lines is the bisector (or pair of bisectors) of the angles formed where the lines cross.

11.5.2 Intersection of Loci: many practical construction problems involve two conditions at once, not just one. Each condition is drawn as its own locus, and the point(s) that satisfy both conditions simultaneously are exactly the point(s) where the two loci intersect. For example, a point that must be 2 cm from a given vertex and also equidistant from two given sides of a triangle is found by drawing a circle of radius 2 cm about the vertex and the angle bisector between the two sides, then marking where the circle crosses the bisector.

11.6 Real-Life Applications of Loci

Loci are not just an abstract geometric exercise — they model many real positioning problems. A fire hydrant or utility pole placed equidistant from two water sources or supply points uses the perpendicular-bisector locus; a robotic arm's safe working zone is often modelled as a circular (or rectangular-band) locus around its base showing every point it can safely reach; GPS receivers and cell-tower networks use circular loci (based on signal-timing distance) and their intersections to pin down a single physical location.

Similarly, an epidemiologist marking a quarantine zone around an outbreak site uses a circular locus of a chosen radius, and land-surveyors or treasure-hunters solving multi-clue positioning puzzles combine several loci — such as 'equidistant from two landmarks' and 'exactly 50 m from a boundary wall' — and locate the unique answer at the intersection of all the given loci.



Important Definitions

Locus

The set of all points in a plane that satisfy one specific given geometric condition or rule (plural: loci).

Perpendicular Bisector (of a segment)

A line that crosses a line segment at its exact midpoint and at a right angle to it.

Median (of a triangle)

A line segment joining a vertex of a triangle to the midpoint of the side opposite that vertex.

Point of Concurrency

A single point at which three or more lines (such as a triangle's medians or bisectors) all meet.

Circumcentre

The point of concurrency of a triangle's three perpendicular bisectors; it is equidistant from all three vertices.

Centroid

The point of concurrency of a triangle's three medians; it always lies inside the triangle and is its balance point.

Incentre

The point of concurrency of a triangle's three angle bisectors; it is equidistant from all three sides.

Orthocentre

The point of concurrency of a triangle's three altitudes.

Angle Bisector (of a triangle's angle)

A ray from a vertex that divides that vertex's interior angle into two exactly equal parts.

Altitude (of a triangle)

A perpendicular line segment drawn from a vertex to the line containing the side opposite that vertex.



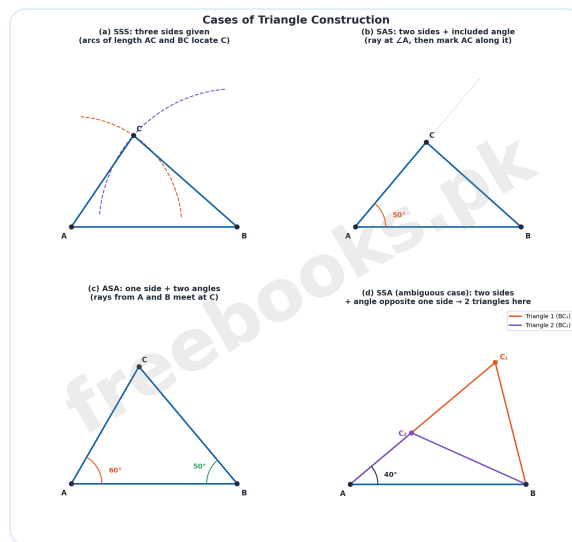
Key Facts and Relations

Topic	Key Fact / Relation
Triangle Inequality Theorem	For any triangle with sides a , b , c : $a + b > c$, $b + c > a$, and $a + c > b$ (the sum of any two sides exceeds the third).
SSA — two triangles	If the given side opposite the angle is greater than the perpendicular distance from the third vertex to the ray, exactly two triangles are possible.
SSA — one triangle (tangent/right-angle case)	If the given side opposite the angle exactly equals the perpendicular distance, exactly one (right-angled) triangle is possible.
SSA — no triangle	If the given side opposite the angle is less than the perpendicular distance, no triangle is possible.
Perpendicular bisectors → Circumcentre	The three perpendicular bisectors of a triangle's sides are concurrent at the circumcentre, equidistant from all three vertices.
Medians → Centroid	The three medians of a triangle are concurrent at the centroid, which always lies inside the triangle.
Angle bisectors → Incentre	The three angle bisectors of a triangle are concurrent at the incentre, equidistant from all three sides.
Altitudes → Orthocentre	The three altitudes of a triangle are concurrent at the orthocentre.
Locus of points equidistant from two intersecting lines	The bisector(s) of the angles formed at the intersection of the two lines.

Diagrams

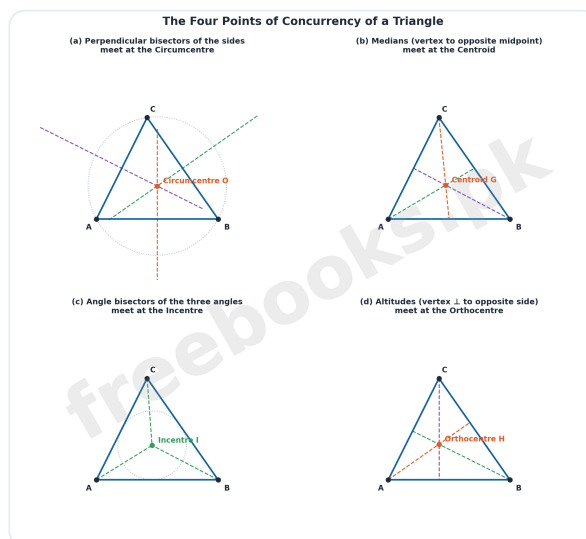
Cases of Triangle Construction: A four-panel figure showing the SSS, SAS, ASA, and SSA (ambiguous) cases of triangle construction, with construction arcs, rays, and labelled vertices





Cases of Triangle Construction

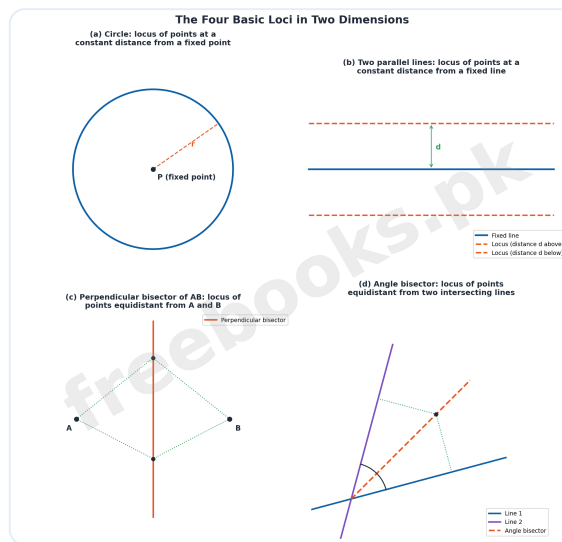
The Four Points of Concurrency of a Triangle: A four-panel figure showing, on the same triangle, the perpendicular bisectors meeting at the circumcentre, the medians meeting at the centroid, the angle bisectors meeting at the incentre, and the altitudes meeting at the orthocentre



The Four Points of Concurrency of a Triangle

The Four Basic Loci in Two Dimensions: A four-panel figure showing the circle locus (fixed point), the parallel-lines locus (fixed line), the perpendicular-bisector locus (two fixed points), and the angle-bisector locus (two intersecting lines)





The Four Basic Loci in Two Dimensions

Solved Examples

Example 1: Constructing a Triangle Given Three Sides (SSS)

Problem: Construct triangle ABC in which $AB = 6$ cm, $BC = 5$ cm, and $CA = 4$ cm.

1. Draw line segment $AB = 6$ cm using a ruler.
2. Open the compass to a radius equal to $CA = 4$ cm, place the point at A, and draw an arc above AB.
3. Open the compass to a radius equal to $BC = 5$ cm, place the point at B, and draw a second arc intersecting the first.
4. Mark the intersection point of the two arcs as C.
5. Join AC and BC with straight lines to complete triangle ABC.

Example 2: Constructing a Triangle Given Two Sides and the Included Angle (SAS)

Problem: Construct triangle PQR in which $PQ = 7$ cm, angle $P = 60^\circ$, and $PR = 5$ cm.

1. Draw line segment $PQ = 7$ cm.



2. At P, construct a ray making an angle of 60° with PQ (using a protractor or a compass-based angle construction).
3. On this ray, mark off a distance of 5 cm from P and label the point R.
4. Join QR with a straight line to complete triangle PQR.
5. Check by measurement that angle P is 60° and that $PR = 5$ cm as required.

Example 3: Investigating the Ambiguous Case (SSA) of Triangle Construction

Problem: In triangle ABC, angle A = 40° , AB = 7 cm, and BC = 5 cm (the side opposite angle A). Determine how many distinct triangles can be constructed, and construct them.

1. Draw AB = 7 cm and construct a ray from A making an angle of 40° with AB.
2. Calculate the perpendicular distance from B to this ray: $BD = AB \times \sin(40^\circ) \approx 7 \times 0.643 \approx 4.5$ cm.
3. Compare the given side BC = 5 cm with this perpendicular distance: since $5 \text{ cm} > 4.5 \text{ cm}$, an arc of radius 5 cm centred at B crosses the ray at two distinct points.
4. Draw an arc of radius 5 cm centred at B; it intersects the ray from A at two points, C_1 and C_2 , each giving a valid triangle.
5. Conclude that two different triangles, ABC_1 and ABC_2 , both satisfy the given SSA data — this is exactly why SSA is called the ambiguous case.



Example 4: Constructing the Perpendicular Bisectors and Medians of a Triangle

Problem: For triangle ABC with $AB = 6$ cm, $BC = 5$ cm, and $CA = 5.5$ cm, construct (a) the three perpendicular bisectors to locate the circumcentre, and (b) the three medians to locate the centroid.

1. Construct triangle ABC using the given side lengths by the SSS method.
2. For each side, open the compass to more than half that side's length and draw arcs above and below the side from both of its endpoints; join the two arc-intersection points to draw the perpendicular bisector of that side.
3. Mark the point where all three perpendicular bisectors meet — this is the circumcentre, equidistant from all three vertices.
4. To find the medians, locate the midpoint of each side (already found while bisecting), then join each midpoint to the vertex opposite it.
5. Mark the point where all three medians meet — this is the centroid, which always lies inside the triangle.

Example 5: Constructing the Angle Bisectors of a Triangle to Locate the Incentre

Problem: For triangle ABC, construct the bisectors of all three interior angles and locate the incentre.

1. At vertex A, draw an arc that cuts both sides AB and AC at equal distances from A.
2. From each of these two intersection points, draw arcs of equal radius that cross inside the angle; join A to this crossing point to obtain the bisector of angle A.
3. Repeat the same process at vertices B and C to construct the bisectors of angle B and angle C.
4. Mark the point where all three angle bisectors meet — this is the incentre, usually labelled I.



5. Note that the incentre is equidistant from all three sides of the triangle, making it the centre of the triangle's inscribed circle (incircle).

Example 6: Constructing the Altitudes of a Triangle to Locate the Orthocentre

Problem: For triangle ABC, construct the three altitudes and locate the orthocentre.

1. From vertex A, draw a perpendicular line to side BC (extending BC if necessary), meeting it at the foot of the altitude.
2. Repeat from vertex B, drawing a perpendicular to side CA.
3. Repeat from vertex C, drawing a perpendicular to side AB.
4. Mark the point where all three altitudes (or their extensions) meet — this is the orthocentre, usually labelled H.
5. Note that in an acute triangle the orthocentre lies inside the triangle, in a right triangle it lies exactly at the right-angled vertex, and in an obtuse triangle it lies outside the triangle.

Example 7: Finding a Point Satisfying Two Loci Conditions

Problem: Point P lies 3 cm from a fixed point O and is also equidistant from two fixed points A and B that are 4 cm apart. Construct the locus of each condition and locate P.

1. Draw a circle of radius 3 cm centred at O — this represents the first locus (all points 3 cm from O).
2. Mark points A and B, 4 cm apart, then construct the perpendicular bisector of AB — this represents the second locus (all points equidistant from A and B).
3. The required point(s) P must lie on both loci simultaneously, so mark where the circle and the perpendicular bisector intersect.



4. If the perpendicular bisector passes within 3 cm of O, it crosses the circle at two points, giving two valid positions for P; if it passes exactly 3 cm from O it touches at one point; otherwise there is no solution.
5. Label the intersection point(s) as the required position(s) of P, and verify by measurement that each is exactly 3 cm from O and equidistant from A and B.

Example 8: Real-Life Application: Locating a Facility Equidistant from Two Roads and a Fixed Distance from a Landmark

Problem: A new fire hydrant must be placed so that it is equidistant from two straight roads that cross at point O, and exactly 6 m from a landmark point L on one of the roads. Determine the possible location(s) for the hydrant.

1. Represent the two roads as two straight lines crossing at O; the locus of points equidistant from both roads is the bisector of the angle they form at O.
2. Construct the angle bisector of the angle between the two roads at O (there are two such bisectors, perpendicular to each other, together covering all four angles at the crossing).
3. Represent the fixed-distance condition as a circle of radius 6 m centred at the landmark point L.
4. Draw the circle of radius 6 m about L, and identify where it crosses the angle-bisector line(s).
5. Each intersection point is a valid hydrant location, since it lies simultaneously on the angle bisector (equidistant from both roads) and on the circle (exactly 6 m from L).

Short Questions & Answers

What is the Triangle Inequality Theorem?



The sum of the lengths of any two sides of a triangle must be greater than the length of the third side; otherwise the three lengths cannot form a triangle.

Why is the SSA case of triangle construction called the ambiguous case?

Because, depending on the given measurements, it can produce two different triangles, exactly one triangle, or no triangle at all, unlike SSS, SAS, and ASA, which always give a unique triangle.

What is the difference between a perpendicular bisector and a median of a triangle?

A perpendicular bisector cuts a side at its midpoint at a right angle and does not necessarily pass through the opposite vertex, while a median joins a vertex directly to the midpoint of the opposite side.

What is a locus?

A locus is the set of all points that satisfy one specific given geometric condition or rule; the plural form is loci.

What shape is the locus of points that are a fixed distance from a fixed point?

A circle, with the fixed point as its centre and the fixed distance as its radius.

What shape is the locus of points equidistant from two fixed points?

The perpendicular bisector of the line segment joining the two fixed points.

What is the locus of points equidistant from two intersecting straight lines?

The bisector (or bisectors) of the angles formed at the intersection of the two lines.

How is the intersection of two loci used to solve a construction problem?

Each locus is drawn as a curve or line representing one condition; any point where the two loci cross satisfies both conditions simultaneously, giving the required point(s).



Long Questions & Answers

Explain the different cases of triangle construction, including the ambiguous case.

How is a triangle constructed when three sides (SSS) are given?

One side is drawn first with a ruler to its exact given length. Then, using a compass, an arc equal to the second given side is drawn from one endpoint of that side, and an arc equal to the third given side is drawn from the other endpoint. The point where the two arcs intersect is the third vertex of the triangle, and joining it to both endpoints of the original side completes the triangle. This method always gives exactly one triangle (up to reflection), provided the three lengths satisfy the Triangle Inequality Theorem.

How is a triangle constructed when two sides and the included angle (SAS) are given?

One of the given sides is drawn first as a straight line segment. At one endpoint of this side, a ray is constructed making the given included angle with the drawn side, using a protractor or a compass-based angle construction. The second given side is then marked off along this new ray from the same endpoint, fixing the position of the third vertex. Joining this new vertex to the far endpoint of the original side completes the triangle uniquely.

How is a triangle constructed when one side and two angles (ASA) are given?

The given side is drawn first between its two endpoints. At each endpoint of this side, a ray is constructed making the respective given angle with the side. Because the two angles are measured on the same side of the base and are less than 180° combined, the two rays are not parallel and must eventually cross at a single point, which becomes the third vertex of the triangle. This intersection point, joined to both original endpoints, completes a uniquely determined triangle.



What makes the SSA case different, and how many triangles can result?

In the SSA case, an angle and two sides are given, but the second side is opposite the given angle rather than included between the two given sides, so its far endpoint is only constrained to lie somewhere on a circular arc rather than at one fixed point. Comparing the given opposite side to the perpendicular distance from the third vertex to the ray from the angle determines the outcome: if the side is longer than this distance, the arc crosses the ray twice, giving two distinct triangles; if it exactly equals the distance, the arc is tangent to the ray, giving exactly one right-angled triangle; and if it is shorter than the distance, the arc never reaches the ray, so no triangle exists at all.

Explain the four types of loci in two dimensions and how the four points of concurrency in a triangle are constructed.

What are the four basic loci in two dimensions?

The four basic loci are: a circle, which is the locus of all points at a fixed distance from a fixed point, drawn by setting a compass to the given distance and swinging it fully around the point; a pair of parallel lines, which is the locus of points at a fixed distance from a fixed straight line, with one line on each side of it; the perpendicular bisector of a segment, which is the locus of all points equidistant from its two endpoints; and the angle bisector, which is the locus of all points equidistant from two intersecting straight lines, found at the bisector(s) of the angles they form.

How are the circumcentre and centroid of a triangle constructed?

The circumcentre is found by constructing the perpendicular bisector of each of the triangle's three sides — each one crossing its side at the midpoint at a right angle — and marking the single point where all three meet; this point is equidistant from all three vertices. The centroid is found by joining each vertex to the midpoint of its opposite side to form the three medians, then marking the single point where all three medians cross; this point always lies inside the triangle and represents its geometric balance point.



How are the incentre and orthocentre of a triangle constructed?

The incentre is found by bisecting each of the triangle's three interior angles with a compass-and-arc construction at each vertex, and marking the single point where all three angle bisectors meet; this point is equidistant from all three sides and is the centre of the triangle's inscribed circle. The orthocentre is found by drawing a perpendicular from each vertex to the line containing the opposite side (extending that side if needed), and marking the point where all three altitudes meet; its position shifts inside, on, or outside the triangle depending on whether the triangle is acute, right-angled, or obtuse.

How is the intersection of two loci used in real-life construction problems?

Real positioning problems usually involve more than one condition at once — for example, a facility that must be a fixed distance from a landmark and also equidistant from two roads. Each condition is drawn as its own locus: a circle for the fixed-distance condition and an angle bisector for the equidistant-from-two-roads condition. Because a valid location must satisfy both rules simultaneously, only the point(s) where the two loci actually cross qualify as solutions, which is why construction problems of this kind are solved by carefully drawing both loci accurately and then reading off their intersection point(s).

Multiple Choice Questions (MCQs)

Which of the following sets of side lengths can form a triangle?

- A. 3 cm, 4 cm, 8 cm
- B. 2 cm, 3 cm, 6 cm
- C. 5 cm, 6 cm, 10 cm
- D. 1 cm, 2 cm, 3 cm

Answer: C. 5 cm, 6 cm, 10 cm

Explanation: By the Triangle Inequality Theorem, $5 + 6 = 11 > 10$, so this set satisfies the condition; the other sets fail it (e.g., $3 + 4 = 7 < 8$).

The three angle bisectors of a triangle are always concurrent at a point called the:

- A. circumcentre
- B. centroid



- C. incentre
- D. orthocentre

Answer: C. incentre

Explanation: The point where all three angle bisectors of a triangle meet is called the incentre, equidistant from all three sides.

The locus of points at a constant distance from a fixed point is a:

- A. straight line
- B. circle
- C. pair of parallel lines
- D. angle bisector

Answer: B. circle

Explanation: A circle is defined as the set of all points at a fixed distance (the radius) from a fixed centre point.

The locus of points equidistant from two fixed points A and B is:

- A. a circle centred at A
- B. a circle centred at B
- C. the perpendicular bisector of AB
- D. a line parallel to AB

Answer: C. the perpendicular bisector of AB

Explanation: Every point on the perpendicular bisector of segment AB is exactly equidistant from A and B.

The locus of points at a constant distance from a fixed straight line is:

- A. a single line parallel to it
- B. two straight lines parallel to it, one on each side
- C. a circle
- D. an angle bisector

Answer: B. two straight lines parallel to it, one on each side

Explanation: Points at a fixed distance from a line exist on both sides of it, forming two parallel lines rather than one.

The locus of points equidistant from two intersecting straight lines is:

- A. the perpendicular bisector of their point of intersection
- B. a circle centred at their intersection
- C. the bisector(s) of the angles between the lines
- D. a line parallel to one of them



Answer: C. the bisector(s) of the angles between the lines

Explanation: Points equidistant from two intersecting lines lie on the bisector(s) of the angles formed at the intersection.

A median of a triangle is a line segment joining:

- A. a vertex to the midpoint of the opposite side
- B. the midpoints of two sides
- C. a vertex perpendicular to the opposite side
- D. two vertices of the triangle

Answer: A. a vertex to the midpoint of the opposite side

Explanation: A median always joins one vertex directly to the midpoint of the side opposite that vertex.

The perpendicular bisectors of the three sides of a triangle meet at the:

- A. centroid
- B. incentre
- C. orthocentre
- D. circumcentre

Answer: D. circumcentre

Explanation: The circumcentre is the point of concurrency of a triangle's three perpendicular bisectors, equidistant from all vertices.

In an obtuse triangle, the orthocentre lies:

- A. inside the triangle
- B. on the triangle
- C. outside the triangle
- D. at the centroid

Answer: C. outside the triangle

Explanation: For an obtuse triangle, the altitudes (or their extensions) meet at a point outside the triangle's boundary.

In the SSA case of triangle construction, if the given side opposite the angle is shorter than the perpendicular distance from the third vertex to the ray, then:

- A. exactly one triangle is possible
- B. exactly two triangles are possible
- C. no triangle is possible
- D. infinitely many triangles are possible



Answer: C. no triangle is possible

Explanation: If the given side cannot reach far enough to meet the ray at all, no triangle can be constructed with the given measurements.

Quick Revision Summary

- A triangle can be constructed uniquely from SSS, SAS, or ASA data using ruler-and-compass methods.
- The SSA case is ambiguous: it can yield two triangles, exactly one triangle (the tangent/right-angle case), or no triangle, depending on how the given opposite side compares with the perpendicular distance from the third vertex.
- The Triangle Inequality Theorem requires the sum of any two sides to exceed the third side for a triangle to exist at all.
- A perpendicular bisector of a side passes through its midpoint at 90° ; the three perpendicular bisectors of a triangle meet at the circumcentre, equidistant from all vertices.
- A median joins a vertex to the midpoint of the opposite side; the three medians meet at the centroid, which always lies inside the triangle.
- An angle bisector splits a vertex angle into two equal parts; the three angle bisectors meet at the incentre, equidistant from all three sides.
- An altitude is a perpendicular from a vertex to the line containing the opposite side; the three altitudes meet at the orthocentre.
- A locus is the set of all points satisfying a single given rule; the four basic 2D loci are a circle, a pair of parallel lines, a perpendicular bisector, and an angle bisector.
- Where two loci intersect, the intersection point(s) satisfy both governing conditions simultaneously.
- Loci have real-life uses such as siting facilities equidistant from two points or roads, defining safe working radii for machinery, and solving land-surveying or multi-clue positioning problems.

Exam Tips

- Always draw a rough sketch of the triangle before starting the accurate ruler-and-compass construction — it helps you place arcs and rays sensibly.



- In SSA problems, calculate the perpendicular distance first (side $\times$ sine of the given angle) and compare it to the given opposite side to know whether to expect 0, 1, or 2 triangles before constructing.
- Keep your compass setting fixed while drawing an arc — do not adjust the radius partway through, or the arc will not be accurate.
- Remember which concurrency point matches which construction:
perpendicular bisectors $\rightarrow$ circumcentre, medians $\rightarrow$ centroid, angle bisectors $\rightarrow$ incentre, altitudes $\rightarrow$ orthocentre.
- When solving locus intersection problems, construct each locus separately and clearly before looking for their intersection — mixing steps causes construction errors.
- Leave all construction arcs visible in your final answer — examiners look for correct compass-arc marks, not just the finished shape, as evidence of proper method.



Unit 12: Information Handling

Information Handling is the branch of statistics concerned with collecting, organising, displaying, and summarising data so that it can be understood and used to draw conclusions. Raw data collected from surveys, experiments, or records is rarely useful on its own — it must first be classified as discrete or continuous, then arranged into a frequency distribution with clearly defined classes, class limits, and class boundaries, before it can be displayed visually as a histogram or a frequency polygon.

Once data has been organised, its most important features can be summarised using measures of central tendency — the arithmetic mean, median, and mode — each of which locates a different kind of 'typical' or 'central' value within the data set. A weighted mean extends the idea of an average to situations where some observations matter more than others. Together, these tools of information handling are used constantly in real life: businesses use them to analyse sales, governments and institutions use them to allocate budgets fairly, and analysts use them to forecast future trends from past data.

Learning Objectives

- Distinguish between discrete data and continuous data, and between ungrouped (raw) data and grouped data.
- Construct a frequency distribution from raw data, including calculating the range, class size, and class boundaries.
- Draw a histogram for grouped data with equal class widths, and for grouped data with unequal class widths using adjusted frequency.
- Draw a frequency polygon from a frequency distribution, both independently and on top of a histogram.
- Calculate the arithmetic mean of grouped and ungrouped data using both the direct method and the short-cut (assumed-mean) method.
- Calculate the median of ungrouped data (odd and even n) and of grouped data using the median formula.



- Calculate the mode of ungrouped data and of grouped data using the mode formula, and calculate a weighted mean.
- Apply measures of central tendency to real-life situations such as sales analysis, fund allocation, and forecasting.

Key Concepts

12.1 Data, Discrete Data, and Continuous Data

Data is any collection of facts, figures, or observations gathered for a particular purpose, while information is the meaningful conclusion drawn after that data has been organised, summarised, and interpreted. Before any conclusion can be drawn, raw data must first be collected, arranged, and analysed using appropriate statistical methods — this entire process is called information handling.

Data is classified into two basic types depending on how it is obtained. Discrete data consists of whole-number values obtained by counting a countable quantity, such as the number of books sold in a shop, the number of students absent from a class, or the number of goals scored in a match — such values cannot take fractional or decimal values (there is no such thing as 3.5 books sold).

Continuous data, by contrast, consists of values obtained by measuring a quantity that can take any value, including decimals, within a range — such as the mass of a student in kilograms, the height of a plant in centimetres, or the time taken to run a race, all of which can be recorded to any required level of precision.

12.2 Ungrouped and Grouped Data; Constructing a Frequency Distribution

Ungrouped data (also called raw data) is data exactly as it was collected, before being organised or arranged in any way — for example, the individual test marks of 30 students listed in the order they were recorded. Grouped data is the same data reorganised into a number of class intervals (or classes), each covering a small range of values, with a count of how many data values fall into each class. Each class has a lower class limit and an upper class limit, the smallest and largest values that the class is defined to contain; a frequency distribution is a



table listing every class alongside its frequency — the number of data values that fall within that class.

A frequency distribution is constructed in a fixed sequence of steps. First, find the range of the raw data: $\text{Range} = X_{\text{max}} - X_{\text{min}}$, the largest value minus the smallest value. Second, decide on a suitable number of classes (commonly 5 to 10) and find the class size by dividing the range by the number of classes, rounding the result up to a convenient whole number: $\text{Class size} = \text{Range} / \text{number of classes}$. Third, set out four columns — class limits, tally marks, frequency, and class boundaries — and tally every raw data value into its correct class to obtain the frequency column. Finally, the class boundaries are found by subtracting half the gap between consecutive class limits (typically 0.5) from every lower limit, and adding the same half-gap to every upper limit, which removes the gap between classes so that the classes run continuously into one another.

12.3 Histogram: Equal and Unequal Class Widths

A histogram is a graph of a frequency distribution in which adjoining rectangular bars are drawn above each class, with no gaps between the bars (since the class boundaries used on the horizontal axis already touch each other). When every class in the distribution has the same width, the histogram is drawn simply by making the height of each bar equal to that class's frequency; taller bars therefore directly represent higher frequencies, and the bars can be compared at a glance.

When the classes of a distribution have unequal widths, however, using the raw frequency as the bar height would be misleading, because a wide class would produce a tall bar purely due to its width, exaggerating its apparent importance compared with a narrow class of similar frequency. To correct for this, each bar's height is instead set to its adjusted frequency, found by dividing the class frequency by the class width: $\text{adjusted frequency} = \text{frequency} / \text{class width}$. Using adjusted frequency as the height ensures that the area of each bar (width $\times$ height) remains proportional to its actual frequency, which is the property a histogram is meant to preserve regardless of how the class widths vary.



12.4 Frequency Polygon

A frequency polygon is a line graph formed by plotting the frequency of each class against its midpoint (also called the class mark), calculated as $\text{midpoint} = (\text{lower class limit or boundary} + \text{upper class limit or boundary}) / 2$, and then joining the plotted points in order with straight line segments. Because it is plotted using midpoints rather than a series of bars, a frequency polygon can be drawn directly from a frequency distribution without first constructing a histogram.

To complete the shape so that it encloses an area comparable to the histogram, the line at each end of the polygon is extended to touch the horizontal axis at the midpoint of the imaginary next class on either side (a class with frequency zero). A frequency polygon can also be drawn on top of an existing histogram — sometimes called a 'frequency polygon on histogram' — by marking the midpoint of the top of each bar and joining these midpoints with straight lines; this variant is useful for comparing the smoothed trend of the data directly against the bars it was derived from.

12.5 Arithmetic Mean

The arithmetic mean, written $\bar{X}$ (read 'x-bar'), is the most commonly used measure of central tendency: it locates the average, or typical central value, of a data set. For ungrouped data, the direct-method formula is $\bar{X} = (\text{Sum of } x) / n$, where the sum of all the individual values is divided by the total number of values, n . For grouped data, each class is represented by its midpoint x , and the direct-method formula becomes $\bar{X} = (\text{Sum of } fx) / (\text{Sum of } f)$, where each midpoint is first multiplied by its class frequency f before the fx values are summed and divided by the total frequency.

When the data values (or midpoints) are large numbers, computing Sum of x or Sum of fx directly can be tedious, so a short-cut (assumed-mean) method is often used instead. An assumed mean A — usually chosen close to the middle of the data — is subtracted from every value (or midpoint) to give a deviation $D = x - A$, which is a much smaller, easier-to-handle number. The mean is then recovered using $\bar{X} = A + (\text{Sum of } D) / n$ for ungrouped data, or $\bar{X} = A +$



$(\text{Sum of } fD) / (\text{Sum of } f)$ for grouped data; both short-cut formulas always give exactly the same answer as the corresponding direct-method formula.

12.6 Median

The median is the middle value of a data set once it has been arranged in ascending (or descending) order, so that exactly half the values lie below it and half lie above it. For ungrouped data with an odd number of values n , the median is the $((n+1)/2)$ th value in the ordered list. For ungrouped data with an even number of values n , there is no single middle value, so the median is taken as the average of the $(n/2)$ th and the $(n/2 + 1)$ th values.

For grouped data, the median is estimated using the formula $\text{Median} = l + (h/f)(n/2 - c)$, where l is the lower class boundary of the median class (the class containing the $(n/2)$ th value), h is the class size, f is the frequency of the median class, n is the total frequency of the whole distribution, and c is the cumulative frequency of all classes before the median class. The median class itself is located by building a cumulative frequency column and finding the first class whose cumulative frequency reaches or exceeds $n/2$.

12.7 Mode, Weighted Mean, and Real-Life Applications

The mode is the value (or class) that occurs most frequently in a data set. Unlike the mean and median, a data set can have no mode at all (if every value occurs equally often), exactly one mode (unimodal), or more than one mode (bimodal or multimodal) if two or more values are tied for the highest frequency. For grouped data, the mode is estimated using $\text{Mode} = l + [(f_m - f_1) / ((f_m - f_1) + (f_m - f_2))] \times h$, where l is the lower boundary of the modal class (the class with the highest frequency f_m), f_1 is the frequency of the class immediately before the modal class, f_2 is the frequency of the class immediately after it, and h is the class size.

The weighted mean, $\bar{X}_w = (\text{Sum of } WX) / (\text{Sum of } W)$, is used whenever different observations carry different levels of importance, represented by a weight W attached to each value X — such as subjects with different credit hours contributing differently to a grade-point average, or items purchased in different quantities contributing differently to an average price. These measures of central tendency have wide real-life use: businesses compute the mean, median, and mode of sales figures to identify typical performance and best-selling items (the



modal class), governments and schools use the mean to allocate funds fairly across departments or sectors based on past spending, and analysts use measures of central tendency together with trends in the data for forecasting and budgeting future needs.

Important Definitions

Data

A collection of facts, figures, or observations gathered for a particular purpose, before it has been organised or interpreted.

Discrete Data

Data consisting of whole-number values obtained by counting a countable quantity, such as the number of items sold; it cannot take fractional or decimal values.

Continuous Data

Data consisting of values obtained by measuring a quantity, which can take any value, including decimals, within a given range, such as mass or height.

Class Limits

The stated lower and upper values that define which raw data values belong to a particular class of a grouped frequency distribution.

Class Boundaries

The true continuous limits of a class, found by adjusting the class limits by half the gap between consecutive classes, so that adjoining classes touch with no gap.

Frequency Distribution

A table that lists every class of a grouped data set alongside its frequency, the number of raw data values that fall within that class.

Histogram

A graph of a frequency distribution made up of adjoining rectangular bars drawn above the class boundaries, with bar height (or adjusted height) representing frequency.

Frequency Polygon



A line graph formed by plotting each class's frequency against its midpoint and joining the points with straight lines, extended to touch the horizontal axis at both ends.

Class Mark (Midpoint)

The middle value of a class, calculated as (lower limit or boundary + upper limit or boundary) / 2; used to represent the whole class in calculations.

Measure of Central Tendency

A single value, such as the mean, median, or mode, that represents or locates the centre of a data set.

Key Facts and Relations

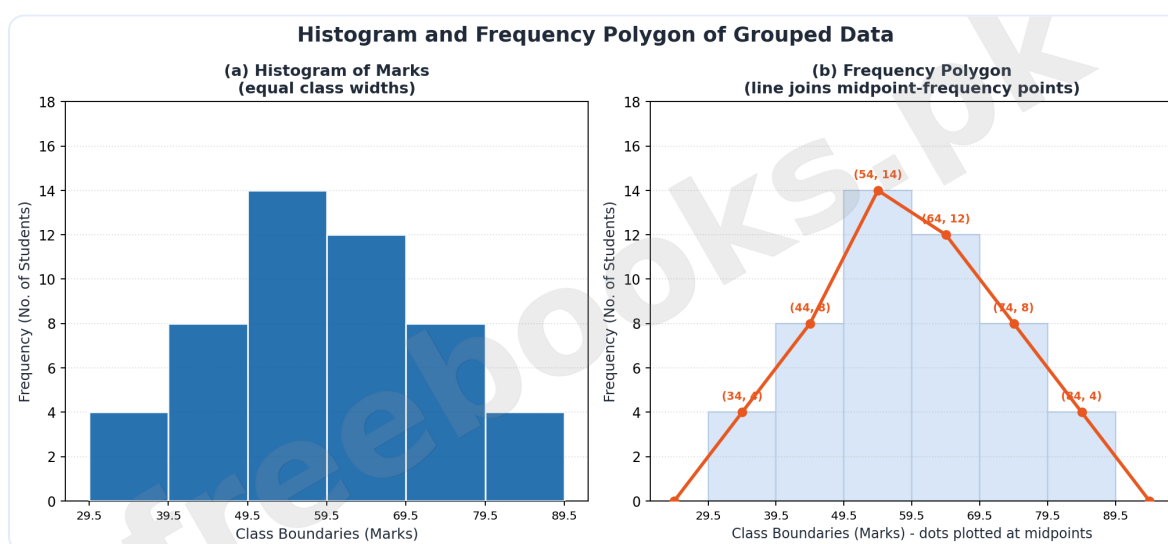
Topic	Key Fact / Relation
Range	Range = $X_{\max} - X_{\min}$, the largest value minus the smallest value in the raw data.
Class Size	Class size = Range / number of classes, rounded up to a convenient whole number.
Class Boundary	Class boundary = class limit $\pm$ half the gap between consecutive class limits (typically ± 0.5).
Arithmetic Mean — direct method	$\bar{X} = (\text{Sum of } x)/n$ (ungrouped); $\bar{X} = (\text{Sum of } fx)/(\text{Sum of } f)$ (grouped, using class midpoints).
Arithmetic Mean — short-cut method	$\bar{X} = A + (\text{Sum of } D)/n$ (ungrouped); $\bar{X} = A + (\text{Sum of } fD)/(\text{Sum of } f)$ (grouped), where $D = x - A$ and A is the assumed mean.
Median — ungrouped data	Odd n : the $((n+1)/2)$ th value; Even n : average of the $(n/2)$ th and $(n/2 + 1)$ th values, once data is arranged in order.
Median — grouped data	Median = $l + (h/f)(n/2 - c)$, where l = lower boundary of median class, h = class size, f = frequency of median class, n = total frequency, c = cumulative frequency before the median class.
Mode — grouped data	Mode = $l + [(f_m - f_1)/((f_m - f_1) + (f_m - f_2))] \times h$, where l = lower boundary of modal class, $f_m/f_1/f_2$ = frequency of the modal, preceding, and following classes, h = class size.



Topic	Key Fact / Relation
Weighted Mean	$\bar{X}_w = (\text{Sum of } WX) / (\text{Sum of } W)$, where W is the weight (importance) attached to each value X .

Diagrams

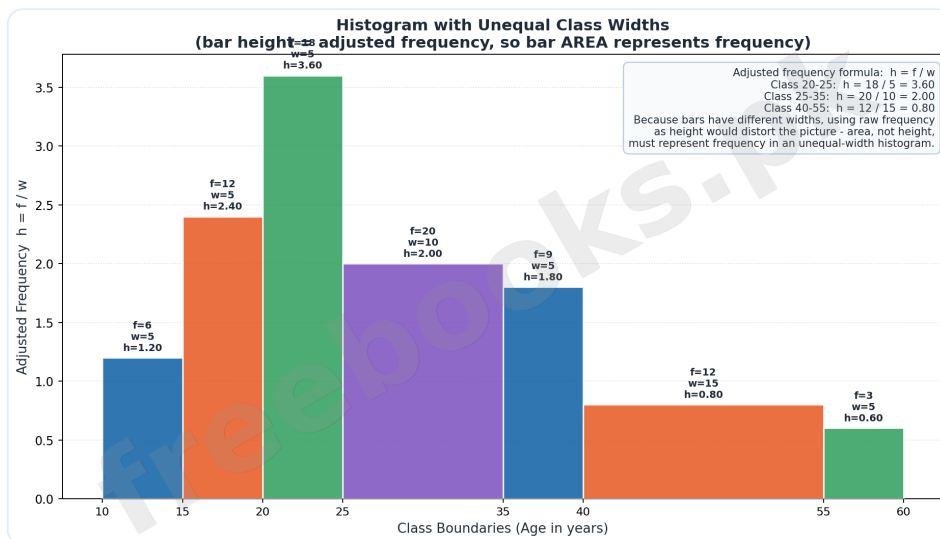
Histogram and Frequency Polygon of Grouped Data: A two-panel figure: the left panel shows a histogram of test marks for 50 students across six equal-width classes; the right panel shows the frequency polygon for the same data, formed by joining midpoint-frequency points and extended to touch the x-axis at both ends.



Histogram and Frequency Polygon of Grouped Data

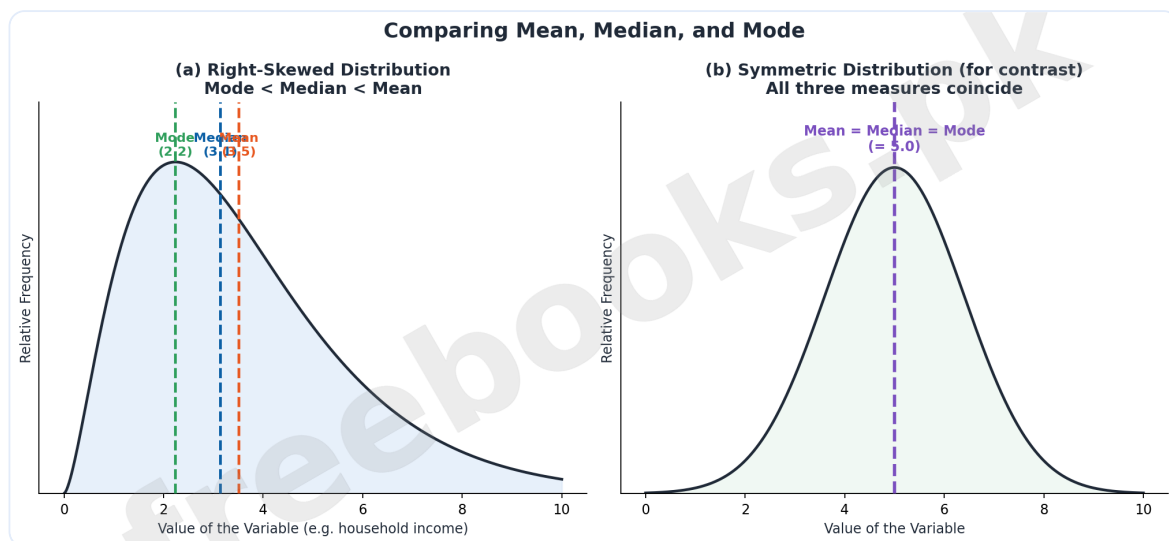
Histogram with Unequal Class Widths: A histogram of the ages of 80 club members grouped into seven classes of different widths, with each bar's height set to the adjusted frequency (frequency / class width) and annotated with the f , w , and h values for each class, so that bar area — not height — represents true frequency.





Histogram with Unequal Class Widths

Comparing Mean, Median, and Mode: A two-panel figure: the left panel shows a right-skewed distribution curve with the mode, median, and mean marked as vertical dashed lines in their typical order (Mode < Median < Mean); the right panel shows a symmetric distribution in which all three measures coincide, for contrast.



Comparing Mean, Median, and Mode



Solved Examples

Example 1: Constructing a Frequency Distribution from Raw Data

Problem: The number of customers served by a small shop on each of 30 days ranges from a minimum of 12 to a maximum of 58 customers. Construct a frequency distribution with 7 classes.

1. Find the range: $\text{Range} = X_{\max} - X_{\min} = 58 - 12 = 46$.
2. Find the class size: $\text{Class size} = \text{Range} / \text{number of classes} = 46 / 7 \approx 6.57$, rounded up to 7 so the classes are wide enough to cover the full range.
3. Starting from the minimum value 12, list the seven classes of width 7: 12-18, 19-25, 26-32, 33-39, 40-46, 47-53, 54-60.
4. Tally the 30 data values into these seven classes to obtain frequencies of 3, 5, 7, 6, 5, 3, and 1, which sum to $3+5+7+6+5+3+1 = 30$, confirming every value was tallied.
5. Find the class boundaries by subtracting 0.5 from each lower limit and adding 0.5 to each upper limit (half the gap of 1 unit between consecutive class limits): 11.5-18.5, 18.5-25.5, 25.5-32.5, 32.5-39.5, 39.5-46.5, 46.5-53.5, 53.5-60.5.

Example 2: Drawing a Histogram with Unequal Class Widths

Problem: The ages of 80 members of a fitness club are grouped into unequal-width classes as follows: 10-15 ($f = 6$), 15-20 ($f = 12$), 20-25 ($f = 18$), 25-35 ($f = 20$), 35-40 ($f = 9$), 40-55 ($f = 12$), 55-60 ($f = 3$). Draw a histogram for this data.

1. Compute the width of each class from its boundaries: 10-15, 15-20, 20-25, 35-40, and 55-60 are each 5 years wide; 25-35 is 10 years wide; 40-55 is 15 years wide.
2. Since the class widths are not equal, using raw frequency as bar height would exaggerate the wider classes, so compute the adjusted frequency $h = f / w$ for each class instead.



3. Class 10-15: $h = 6/5 = 1.2$. Class 15-20: $h = 12/5 = 2.4$. Class 20-25: $h = 18/5 = 3.6$.
4. Class 25-35: $h = 20/10 = 2.0$. Class 35-40: $h = 9/5 = 1.8$.
5. Class 40-55: $h = 12/15 = 0.8$. Class 55-60: $h = 3/5 = 0.6$. Draw bars on the class-boundary x-axis with these adjusted-frequency heights, so each bar's area (width $\times$ height) equals its true frequency.

Example 3: Drawing a Frequency Polygon

Problem: Marks obtained by 50 students in a mathematics test are grouped as follows (class boundaries and frequency): 29.5-39.5 ($f=4$), 39.5-49.5 ($f=8$), 49.5-59.5 ($f=14$), 59.5-69.5 ($f=12$), 69.5-79.5 ($f=8$), 79.5-89.5 ($f=4$). Draw the frequency polygon for this distribution.

1. Find the midpoint (class mark) of each class using midpoint = (lower + upper)/2: 34.5, 44.5, 54.5, 64.5, 74.5, 84.5.
2. Plot each pair (midpoint, frequency): (34.5, 4), (44.5, 8), (54.5, 14), (64.5, 12), (74.5, 8), (84.5, 4).
3. Join the plotted points in order with straight line segments to form the body of the polygon.
4. Extend the line at each end to touch the x-axis at the midpoint of the next (empty) class on either side: 24.5 before the first class and 94.5 after the last class, each with frequency 0.
5. Check that the frequencies sum to the total number of students:
 $4+8+14+12+8+4 = 50$, matching the given 50 students.

Example 4: Arithmetic Mean by the Direct Method (Grouped Data)

Problem: The weights (in kg) of 40 parcels handled by a courier office in one day are grouped as follows: 1-5 kg ($f=4$), 6-10 kg ($f=10$), 11-15 kg ($f=14$), 16-20 kg ($f=8$), 21-25 kg ($f=4$). Find the arithmetic mean weight using the direct method.

1. Find the midpoint x of each class: 3, 8, 13, 18, 23.



2. Multiply each midpoint by its frequency (fx): $4 \times 3 = 12$, $10 \times 8 = 80$, $14 \times 13 = 182$, $8 \times 18 = 144$, $4 \times 23 = 92$.
3. Sum the frequencies: Sum of f = $4 + 10 + 14 + 8 + 4 = 40$.
4. Sum the fx values: Sum of fx = $12 + 80 + 182 + 144 + 92 = 510$.
5. Apply the direct-method formula: $\bar{X} = (\text{Sum of fx}) / (\text{Sum of f}) = 510 / 40 = 12.75 \text{ kg}$.

Example 5: Arithmetic Mean by the Short-Cut (Assumed-Mean) Method

Problem: Using the same parcel-weight data as the previous example (classes with midpoints 3, 8, 13, 18, 23 and frequencies 4, 10, 14, 8, 4), find the arithmetic mean using the short-cut formula, taking the assumed mean $A = 13$.

1. Take the assumed mean $A = 13$ (the midpoint of the middle class).
2. Find the deviation $D = x - A$ for each class: $3 - 13 = -10$, $8 - 13 = -5$, $13 - 13 = 0$, $18 - 13 = 5$, $23 - 13 = 10$.
3. Multiply each deviation by its frequency (fD): $4 \times (-10) = -40$, $10 \times (-5) = -50$, $14 \times 0 = 0$, $8 \times 5 = 40$, $4 \times 10 = 40$.
4. Sum the fD values: Sum of fD = $-40 + (-50) + 0 + 40 + 40 = -10$.
5. Apply the short-cut formula: $\bar{X} = A + (\text{Sum of fD}) / (\text{Sum of f}) = 13 + (-10/40) = 13 - 0.25 = 12.75 \text{ kg}$, exactly matching the direct-method result.

Example 6: Median for Grouped Data

Problem: Using the test-marks distribution (classes 29.5-39.5 to 79.5-89.5 with frequencies 4, 8, 14, 12, 8, 4, total $n=50$), find the median mark.

1. Find $n/2 = 50/2 = 25$.
2. Build the cumulative frequency column: 4, 12, 26, 38, 46, 50.



3. Identify the median class as the first class whose cumulative frequency is ≥ 25 : this is 49.5-59.5, with cumulative frequency 26.
4. Note the values needed: $l = 49.5$ (lower boundary), $h = 10$ (class size), $f = 14$ (frequency of median class), $c = 12$ (cumulative frequency before the median class).
5. Apply the median formula: $\text{Median} = l + (h/f)(n/2 - c) = 49.5 + (10/14)(25-12) = 49.5 + (10/14)(13) \approx 49.5 + 9.29 = 58.79$ marks.

Example 7: Mode for Grouped Data

Problem: Using the same test-marks distribution, the modal class has the highest frequency of 14, occurring in class 49.5-59.5, with the classes immediately before and after having frequencies 8 and 12 respectively. Find the mode.

1. Identify the modal class as the class with the highest frequency: 49.5-59.5, with $f_m = 14$.
2. Note the frequency of the class before the modal class: $f_1 = 8$ (class 39.5-49.5).
3. Note the frequency of the class after the modal class: $f_2 = 12$ (class 59.5-69.5).
4. Note $l = 49.5$ (lower boundary of modal class) and $h = 10$ (class size).
5. Apply the mode formula: $\text{Mode} = l + [(f_m - f_1) / ((f_m - f_1) + (f_m - f_2))] \times h = 49.5 + [(14 - 8) / ((14 - 8) + (14 - 12))] \times 10 = 49.5 + (6/8) \times 10 = 49.5 + 7.5 = 57$ marks.

Example 8: Weighted Mean

Problem: A student's exam marks and the credit-hour weight of each subject are: Mathematics 85 (weight 4), Physics 78 (weight 3), Chemistry 82 (weight



3), English 75 (weight 2), Computer Science 90 (weight 2). Find the weighted mean mark.

1. Multiply each mark by its corresponding weight (WX): $85 \times 4 = 340$, $78 \times 3 = 234$, $82 \times 3 = 246$, $75 \times 2 = 150$, $90 \times 2 = 180$.
2. Sum the WX values: Sum of WX = $340 + 234 + 246 + 150 + 180 = 1150$.
3. Sum the weights: Sum of W = $4 + 3 + 3 + 2 + 2 = 14$.
4. Apply the weighted mean formula: $\bar{X}_w = (\text{Sum of WX}) / (\text{Sum of W}) = 1150 / 14 \approx 82.14$.
5. Compare with the simple (unweighted) mean of the marks, $(85 + 78 + 82 + 75 + 90) / 5 = 410 / 5 = 82.0$, showing the weighted mean shifts slightly toward the higher-weight subject (Mathematics).

Short Questions & Answers

What is the difference between discrete data and continuous data? Give one example of each.

Discrete data consists of whole-number values obtained by counting, such as the number of cars sold in a showroom in a month, and cannot take fractional values. Continuous data consists of values obtained by measuring, such as the height of students in centimetres, and can take any value including decimals within a range.

What is the difference between ungrouped (raw) data and grouped data?

Ungrouped (raw) data is data exactly as collected, listed individually without any organisation. Grouped data is the same data reorganised into a number of class intervals, with the number of values falling into each class recorded as its frequency.

How are class boundaries calculated from class limits?

Class boundaries are found by subtracting half the gap between consecutive class limits (usually 0.5) from every lower class limit, and adding the same half-gap to every upper class limit. This removes the gap between classes so that



consecutive classes touch each other continuously, which is needed for drawing an accurate histogram.

Why is bar height adjusted (using f/w) in a histogram with unequal class widths?

If classes have unequal widths, using raw frequency as bar height would make wider classes look taller and more important than they really are. Dividing frequency by class width (f/w) gives an adjusted frequency, so that the area of each bar (width $\times$ height) stays proportional to its true frequency.

How is a frequency polygon constructed from a frequency distribution?

Plot the midpoint (class mark) of each class against its frequency, then join these points in order with straight line segments. The line is extended at both ends to touch the horizontal axis at the midpoints of the imaginary empty classes just before the first and just after the last class.

What is the median class in grouped data, and how is it identified?

The median class is the class that contains the $(n/2)$ th value of the data set. It is identified by building a cumulative frequency column and finding the first class whose cumulative frequency is equal to or greater than $n/2$, where n is the total frequency.

What is a weighted mean and when should it be used instead of a simple arithmetic mean?

A weighted mean, $\bar{X}_w = (\text{Sum of } WX) / (\text{Sum of } W)$, gives each value X an importance weight W before averaging. It should be used instead of a simple mean whenever observations are not all equally important, such as subjects with different credit hours or items bought in different quantities.

Can a set of data have more than one mode? Explain.

Yes. A data set is unimodal if exactly one value has the highest frequency, but it can be bimodal or multimodal if two or more values are tied for the highest frequency, or it can have no mode at all if every value occurs the same number of times.



Long Questions & Answers

Explain how a frequency distribution is constructed from raw data, including histograms with equal and unequal class widths.

What are the steps to construct a frequency distribution from raw (ungrouped) data?

Constructing a frequency distribution begins by finding the range of the raw data, $\text{Range} = X_{\text{max}} - X_{\text{min}}$, the difference between the largest and smallest values. A suitable number of classes (commonly 5 to 10) is chosen, and the class size is found by dividing the range by the number of classes and rounding up: $\text{Class size} = \text{Range} / \text{number of classes}$. A table is then set out with four columns — class limits, tally marks, frequency, and class boundaries — and every raw data value is tallied into its correct class to build up the frequency column. Finally, class boundaries are calculated by adjusting the class limits by half the gap between consecutive classes.

What are class limits and class boundaries, and how do they differ?

Class limits are the stated lower and upper values that define which raw data values belong to a class, for example the class '11-20' has a lower limit of 11 and an upper limit of 20. Because consecutive class limits usually leave a small gap (such as the gap between 20 and the next class's lower limit of 21), class limits alone cannot be used directly to draw an accurate histogram. Class boundaries close this gap: half of the gap between consecutive limits (typically 0.5) is subtracted from each lower limit and added to each upper limit, giving continuous boundaries such as 10.5-20.5, which touch the next class's boundary exactly.

How is a histogram drawn when all classes have equal width?

When every class in a frequency distribution has the same width, a histogram is drawn by marking the class boundaries along the horizontal axis and the frequency scale along the vertical axis, then drawing a rectangular bar above each class whose height is simply equal to that class's frequency. Because all the bars share the same width, taller bars directly and fairly represent higher



frequencies, and no adjustment to the height is needed. The bars are drawn touching each other, with no gaps between them, since the class boundaries of adjoining classes are identical.

How is a histogram drawn when classes have unequal width, and why is adjustment needed?

When classes have unequal widths, using the raw frequency as bar height would make wider classes appear taller and more significant purely because of their width, distorting the true picture of the data. To avoid this, each bar's height is set to its adjusted frequency, calculated as $\text{adjusted frequency} = \text{frequency} / \text{class width}$. This ensures that the area of each bar, not just its height, is what represents the class's actual frequency, keeping the histogram's proportions accurate and comparable even though the bars have different widths.

Explain the three measures of central tendency (mean, median, mode) and weighted mean, including their formulas for grouped data.

How is the arithmetic mean of grouped data calculated by the direct method and the short-cut (assumed-mean) method?

By the direct method, each class is represented by its midpoint x , which is multiplied by the class frequency f to give fx ; these fx values are summed and divided by the total frequency, giving $\bar{X} = (\text{Sum of } fx) / (\text{Sum of } f)$. By the short-cut (assumed-mean) method, a convenient assumed mean A near the centre of the data is chosen, and the deviation $D = x - A$ is found for each class; the mean is then $\bar{X} = A + (\text{Sum of } fD) / (\text{Sum of } f)$. Both methods always give the identical final answer, but the short-cut method involves smaller, easier numbers when the original values are large.

How is the median of grouped data calculated, and what does each symbol in the formula represent?

The median of grouped data is calculated using $\text{Median} = l + (h/f)(n/2 - c)$. Here, l is the lower class boundary of the median class, the class containing the $(n/2)$ th value; h is the class size (width) of the median class; f is the frequency of the median class; n is the total frequency of the whole distribution; and c is the



cumulative frequency of all the classes that come before the median class. The median class itself is located first, by building a cumulative frequency column and finding the first class whose cumulative frequency reaches or exceeds $n/2$.

How is the mode of grouped data calculated, and what does each symbol in the formula represent?

The mode of grouped data is calculated using $\text{Mode} = l + \frac{(f_m - f_1)}{(f_m - f_1) + (f_m - f_2)} \times h$. Here, l is the lower class boundary of the modal class, the class with the highest frequency; f_m is the frequency of the modal class itself; f_1 is the frequency of the class immediately before the modal class; f_2 is the frequency of the class immediately after it; and h is the class size. The formula works by weighting the modal class boundary toward whichever neighbouring class is more crowded, giving a more accurate estimate than simply reporting the midpoint of the modal class.

What is a weighted mean, and how does it differ from a simple arithmetic mean in real-life applications?

A weighted mean, $\bar{X}_w = (\text{Sum of } WX) / (\text{Sum of } W)$, multiplies each value X by an importance weight W before averaging, then divides by the sum of the weights rather than by a plain count of values. This differs from a simple arithmetic mean, which treats every value as equally important. In real life, a student's grade-point average uses the credit hours of each subject as weights so that a heavier subject contributes more to the final average, and a shopkeeper computing an average price per item across different products uses the quantity sold of each product as its weight.

Multiple Choice Questions (MCQs)

The number of books sold by a shop each day is an example of:

- A. Continuous data
- B. Discrete data
- C. Grouped data
- D. Weighted data

Answer: B. Discrete data



Explanation: The number of books sold is a whole-number value obtained by counting, so it is discrete data; it cannot take fractional values.

The number of times a value occurs in a data set is called its:

- A. Class boundary
- B. Class limit
- C. Frequency
- D. Range

Answer: C. Frequency

Explanation: Frequency is defined as the number of times a value, or the number of data values within a class, occurs in a data set.

The midpoint of a class is also known as its:

- A. Class boundary
- B. Class limit
- C. Class mark
- D. Class range

Answer: C. Class mark

Explanation: The midpoint of a class, calculated as $(\text{lower} + \text{upper})/2$, is also called the class mark, and represents the whole class in calculations.

Which of the following statements about a frequency polygon is true?

- A. It can only be drawn on top of an already-drawn histogram
- B. It is formed by joining the midpoint-frequency points of a frequency distribution with straight lines, and can be drawn with or without a histogram
- C. It uses class limits instead of midpoints
- D. It is only used for continuous data

Answer: B. It is formed by joining the midpoint-frequency points of a frequency distribution with straight lines, and can be drawn with or without a histogram

Explanation: A frequency polygon is plotted directly from midpoints and frequencies, so it can be drawn on its own or on top of a histogram.

The range of a data set is calculated as:

- A. $X_{\text{max}} + X_{\text{min}}$
- B. $X_{\text{max}} - X_{\text{min}}$
- C. $(X_{\text{max}} + X_{\text{min}}) / 2$
- D. $X_{\text{max}} \times X_{\text{min}}$

Answer: B. $X_{\text{max}} - X_{\text{min}}$



Explanation: The range is the difference between the largest value ($X_{\max}$) and the smallest value ($X_{\min}$) in the data set.

A measure of central tendency describes:

- A. The spread of the data
- B. A value at or near the centre of the data
- C. The largest value in the data
- D. The number of classes in the data

Answer: B. A value at or near the centre of the data

Explanation: Measures of central tendency such as the mean, median, and mode locate a typical or central value within a data set.

If the mean of the data set 4, 6, x, 10, 12 is 8, what is the value of x?

- A. 6
- B. 7
- C. 8
- D. 9

Answer: C. 8

Explanation: Sum of values = $5 \times 8 = 40$, so $4+6+x+10+12 = 40$, giving $32+x = 40$, so $x = 8$.

The mode of the data set 3, 5, 5, 7, 8, 5, 9 is:

- A. 3
- B. 5
- C. 7
- D. 9

Answer: B. 5

Explanation: The value 5 occurs three times, more often than any other value, so the mode is 5.

A data set can have:

- A. Only one mode
- B. No mode, one mode, or more than one mode
- C. Always exactly two modes
- D. No mode under any circumstances

Answer: B. No mode, one mode, or more than one mode

Explanation: Depending on how frequencies are distributed, a data set may have zero, one, or several modes (unimodal, bimodal, or multimodal).



The median of the data set 12, 7, 15, 9, 21 is:

- A. 9
- B. 12
- C. 15
- D. 21

Answer: B. 12

Explanation: Arranged in order the data is 7, 9, 12, 15, 21; with $n=5$ (odd), the median is the middle (3rd) value, which is 12.

Quick Revision Summary

- Discrete data is obtained by counting and takes whole-number values; continuous data is obtained by measuring and can take any value, including decimals.
- Ungrouped (raw) data is listed individually; grouped data is organised into class intervals with a recorded frequency for each class.
- Range = $X_{\max} - X_{\min}$; Class size = Range / number of classes, rounded up to a convenient whole number.
- Class boundaries are found by adjusting class limits by half the gap between consecutive classes (typically 0.5), removing gaps so classes touch continuously.
- In a histogram with equal class widths, bar height = frequency; with unequal class widths, bar height = adjusted frequency = frequency / class width, so bar area represents frequency.
- A frequency polygon joins (midpoint, frequency) points with straight lines and is extended to touch the x-axis at the midpoints of the empty classes on either end.
- Arithmetic mean: $\bar{X} = (\text{Sum of } fx) / (\text{Sum of } f)$ by the direct method, or $\bar{X} = A + (\text{Sum of } fD) / (\text{Sum of } f)$ by the short-cut method, where $D = x - A$.
- Median (grouped) = $l + (h/f)(n/2 - c)$; Mode (grouped) = $l + [(f_m - f_1) / ((f_m - f_1) + (f_m - f_2))] \times h$.
- A data set can have zero, one, or more than one mode; the weighted mean $\bar{X}_w = (\text{Sum of } WX) / (\text{Sum of } W)$ is used when observations carry different importance.



- Measures of central tendency are widely used in real life for sales analysis, fund allocation across sectors, and forecasting and budgeting.

Exam Tips

- Always double-check that your tallied frequencies add up to the total number of raw data values before drawing any graph.
- Remember class boundaries, not class limits, are used on the horizontal axis of a histogram, so classes touch with no gaps.
- In an unequal-width histogram, never use raw frequency as bar height — always compute adjusted frequency = frequency / class width first.
- When finding the median or mode of grouped data, build the cumulative frequency column carefully — a single tallying mistake will shift the whole answer.
- Verify a short-cut (assumed-mean) calculation by checking that it matches the direct-method mean; both must give exactly the same result.
- In weighted mean problems, always multiply each value by its own weight before summing — do not average the weights and values separately.



Unit 13: Probability

Probability is the branch of mathematics that measures the chance of occurrence of a particular event, turning everyday ideas like 'likely', 'unlikely', or 'impossible' into precise numbers between 0 and 1. It grew out of the study of games of chance — tossing coins, rolling dice, drawing cards — but its methods now underpin weather forecasting, insurance and risk assessment, quality control in factories, medical testing, and countless other real-life decisions made under uncertainty.

At the heart of probability lies a simple formula, $P(A) = n(A)/n(S)$, which compares the number of favourable outcomes of an event to the total number of possible outcomes in the sample space. Building on this single idea, the complement rule lets us find the probability that an event does NOT happen, while relative frequency and expected frequency extend probability thinking to real recorded data — surveys, experiments, and production records — allowing predictions to be made about how often an event should occur in a large number of trials.

Learning Objectives

- Define an experiment, outcome, favourable outcome, sample space, event, and sample point, with suitable examples.
- State and apply the basic formula for probability, $P(A) = n(A)/n(S)$, and explain why $0 \leq P(A) \leq 1$.
- Identify certain, impossible, likely, unlikely, and equally likely events, and state their probabilities.
- Find the probability of a single event using a single die and using the 36-outcome sample space of two dice.
- State and apply the complement rule, $P(A') = 1 - P(A)$, to find the probability of an event not occurring.
- Apply probability concepts to real-life problems involving targets, bags of balls, and playing cards.



- Define relative frequency, calculate it from a frequency table, and use it as an estimate of probability.
- Define expected frequency, calculate it using $\text{Expected frequency} = N \times P(A)$, and apply it to real-life situations.

Key Concepts

13.0 Introduction to Probability

Probability is a way of measuring the chance of occurrence of a particular event, expressed as a number between 0 and 1. An event that is certain to happen has a probability of 1, an event that can never happen has a probability of 0, and every other event falls somewhere between these two extremes depending on how likely it is.

The probability of an event A is defined by the formula $P(A) = n(A)/n(S)$, where $n(A)$ is the number of favourable outcomes — the outcomes that satisfy the event A — and $n(S)$ is the total number of possible outcomes in the sample space S . This single formula is the foundation on which every other idea in this unit, including the complement rule, relative frequency, and expected frequency, is built.

13.1 Basic Concepts of Probability

An experiment is any process that generates a result, such as tossing a coin, rolling a die, or drawing a card from a deck. Each individual result produced by an experiment is called an outcome, and an outcome that matches the particular result we are interested in is called a favourable outcome. The sample space S is the set of all possible outcomes of an experiment — for tossing a single coin, $S = \{H, T\}$; for rolling a single die, $S = \{1, 2, 3, 4, 5, 6\}$. Each individual member of the sample space is called a sample point.

An event is any subset of the sample space — a collection of one or more outcomes that we are interested in. For example, if a die is rolled and A is the event 'an even number appears', then $A = \{2, 4, 6\}$ and $n(A) = 3$. Events are classified by how likely they are: a certain event is one that must happen, with $P(A) = 1$ (for example, getting a number from 1 to 6 on a die); an impossible



event can never happen, with $P(A) = 0$ (for example, getting a 7 on an ordinary die); a likely event has a probability greater than 0.5, an unlikely event has a probability less than 0.5, and an equally likely event has a probability of exactly 0.5, such as getting heads on a single coin toss. For any event A , the probability always satisfies $0 \leq P(A) \leq 1$.

13.2 Probability of a Single Event

For an experiment with a single die, the sample space is $S = \{1, 2, 3, 4, 5, 6\}$, so $n(S) = 6$. To find the probability of any event, we count the favourable outcomes within this sample space and divide by 6 — for example, the probability of getting a number divisible by 3 is $n(A)/n(S) = 2/6 = 1/3$, since $A = \{3, 6\}$.

When two dice are rolled together, every outcome is an ordered pair (first die, second die), giving a sample space of $n(S) = 6 \times 6 = 36$ equally likely outcomes, often displayed as a 6-by-6 grid. Events involving two dice are found the same way — by counting how many of the 36 pairs satisfy the event. For example, the event 'both dice show an even number' contains the pairs where both coordinates are from $\{2, 4, 6\}$, giving $n(A) = 3 \times 3 = 9$ outcomes, so $P(A) = 9/36 = 1/4$; similarly, the event 'the sum of the two dice equals 8' contains the pairs $(2,6), (3,5), (4,4), (5,3), (6,2)$, giving $n(A) = 5$, so $P(A) = 5/36$.

13.3 Probability of an Event Not Occurring

The complement of an event A , written A' , is the event that A does NOT occur — it consists of every outcome in the sample space that is not in A . Since every outcome in S either belongs to A or to A' but never both, the two probabilities must add up to exactly 1: $P(A) + P(A') = 1$. Rearranging this relationship gives the complement rule, $P(A') = 1 - P(A)$, which lets us find the probability of an event not happening directly from the probability of it happening, without recounting outcomes.

The complement rule is especially useful when the 'not occurring' event is easier to describe than a long list of favourable outcomes. For example, the probability of not getting a 6 on a single die is $P(A') = 1 - P(6) = 1 - 1/6 = 5/6$; the probability of not getting a double-six when two dice are thrown together is $P(A') = 1 - 1/36 = 35/36$, since only one of the 36 outcomes, $(6,6)$, is a double six.



13.4 Real-Life Problems Involving Probability

Probability, together with the complement rule, is regularly used to analyse real-life situations that involve chance. For example, if a missile has a known probability of hitting a target, the probability of it missing the target is found using the complement rule; if a bag contains a known number of blue and green balls, the probability of drawing 'not green' can be found either by counting the blue balls directly or by subtracting $P(\text{green})$ from 1.

Playing cards provide another common real-life context: a standard deck has 52 cards divided into 4 suits (hearts, diamonds, clubs, spades) of 13 cards each. The probability of drawing a heart is $P(\text{heart}) = 13/52 = 1/4$, and since spades and hearts together make up 26 of the 52 cards, $P(\text{spade or heart}) = 26/52 = 1/2$; the complement rule then gives the probability of drawing neither a spade nor a heart as $1 - 1/2 = 1/2$.

13.5 Relative Frequency as an Estimate of Probability

While the formula $P(A) = n(A)/n(S)$ works well for simple experiments like dice and coins where every outcome is equally likely, many real-life situations involve recorded data instead — surveys, experiments repeated many times, or production records — where relative frequency is used as a practical estimate of probability. The relative frequency of an event is defined as $\text{Relative frequency} = (\text{frequency of the specific event}) / (\text{total frequency}) = x/N$, where x is how many times the event occurred and N is the total number of observations (Sum of f).

Relative frequency behaves like a probability: it always lies between 0 and 1, and when the relative frequencies of every category in a complete frequency distribution are added together, the sum is always approximately 1 (exactly 1 if no rounding has occurred). This makes relative frequency a natural bridge between recorded real-world data and the theoretical idea of probability — as the number of observations N grows very large, the relative frequency of an event approaches its true theoretical probability.

13.6 Expected Frequency and Its Real-Life Application

Expected frequency turns the idea of probability around: instead of using data to estimate a probability, it uses a known (or assumed) probability to predict how



many times an event should occur in a given number of trials. The expected frequency of an event is calculated as $\text{Expected frequency} = \text{Total number of trials} \times \text{Probability of the event} = N \times P(A)$. Just as the relative frequencies of a complete distribution sum to approximately 1, the expected frequencies of all the categories in a distribution sum to approximately N , the fixed total number of trials.

Expected frequency has wide real-life application wherever a known probability needs to be converted into a predicted count. For example, if a fair die is rolled 300 times, the expected number of times a 1 or a 6 appears is $N \times P(A) = 300 \times (2/6) = 100$; if a factory knows from experience that the probability of a manufactured bolt being defective is 0.3, then out of 800 bolts produced, the expected number of non-defective bolts is $800 \times 0.7 = 560$, information a quality-control team can use to plan inspection and replacement in advance.

Important Definitions

Experiment

Any process that generates a result or outcome, such as tossing a coin, rolling a die, or drawing a card from a deck.

Sample Space (S)

The set of all possible outcomes of an experiment; for a single die, $S = \{1, 2, 3, 4, 5, 6\}$, so $n(S) = 6$.

Sample Point

Each individual member (outcome) of a sample space; for example, the outcome '4' is one sample point in the sample space of a die.

Event

Any subset of the sample space — a collection of one or more outcomes that satisfy a particular condition of interest.

Favourable Outcome

An outcome of an experiment that matches or satisfies the particular event being considered.

Certain Event



An event that is sure to happen every time the experiment is performed, with probability $P(A) = 1$.

Impossible Event

An event that can never happen, no matter how many times the experiment is performed, with probability $P(A) = 0$.

Complement of an Event (A')

The event that A does NOT occur, consisting of every outcome in the sample space that is not in A; $P(A') = 1 - P(A)$.

Relative Frequency

An estimate of probability calculated from recorded data as (frequency of the event) / (total frequency), written x/N .

Expected Frequency

The predicted number of times an event should occur in N trials, calculated as $N \times P(A)$; also called theoretical frequency.

Key Facts and Relations

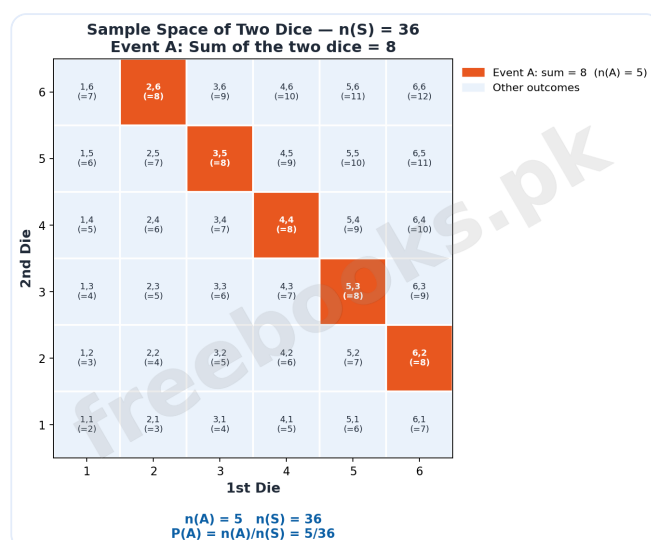
Topic	Key Fact / Relation
Probability of an Event	$P(A) = n(A)/n(S)$, where $n(A)$ = number of favourable outcomes, $n(S)$ = total number of possible outcomes.
Range of Probability	$0 \leq P(A) \leq 1$ for every event A; $P(A) = 0$ means impossible, $P(A) = 1$ means certain.
Complement Rule	$P(A') = 1 - P(A)$, the probability of an event NOT occurring equals 1 minus the probability of it occurring.
Event and Its Complement	$P(A) + P(A') = 1$, since every outcome of S belongs either to A or to A' , but never both.
Sample Space Size for n Dice	$n(S) = 6^n$ when n fair dice are rolled together; for two dice, $n(S) = 6^2 = 36$.
Relative Frequency	Relative frequency = x/N , where x = frequency of the specific event, N = total frequency (Sum of f).



Topic	Key Fact / Relation
Sum of Relative Frequencies	The relative frequencies of every category in a complete distribution sum to approximately 1.
Expected Frequency	Expected frequency = $N \times P(A)$, where N = total number of trials and $P(A)$ = probability of the event.
Sum of Expected Frequencies	The expected frequencies of every category in a distribution sum to approximately N , the fixed number of trials.

Diagrams

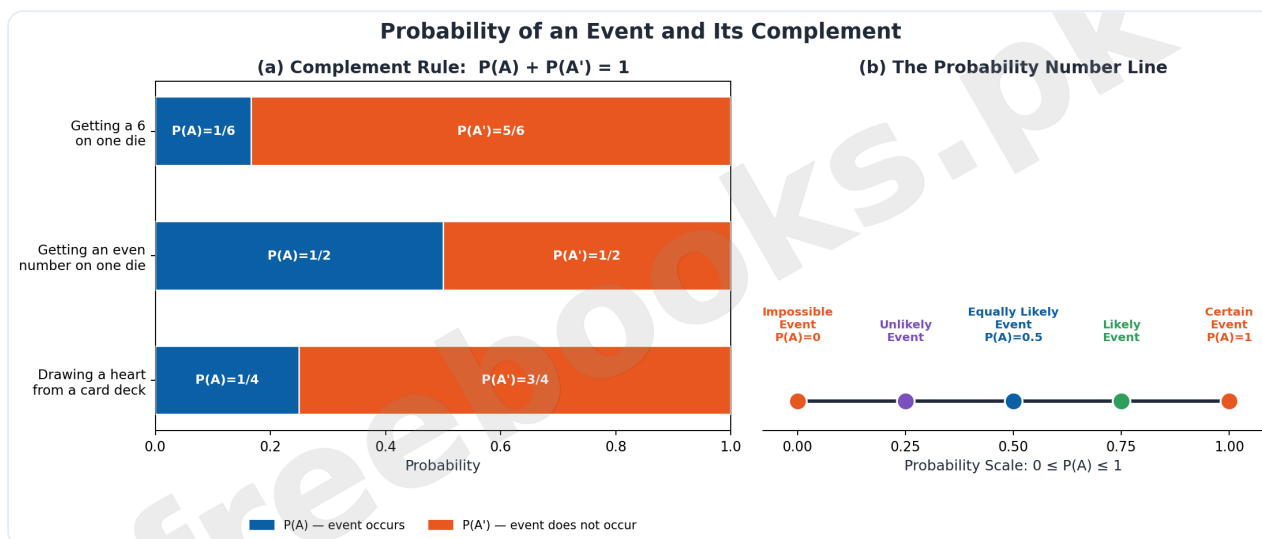
Sample Space of Two Dice: A 6x6 grid showing all 36 equally likely outcomes of rolling two fair dice, with the cells where the sum of the two dice equals 8 highlighted in orange ($n(A) = 5$), illustrating how to count favourable outcomes for a two-dice event.



Sample Space of Two Dice

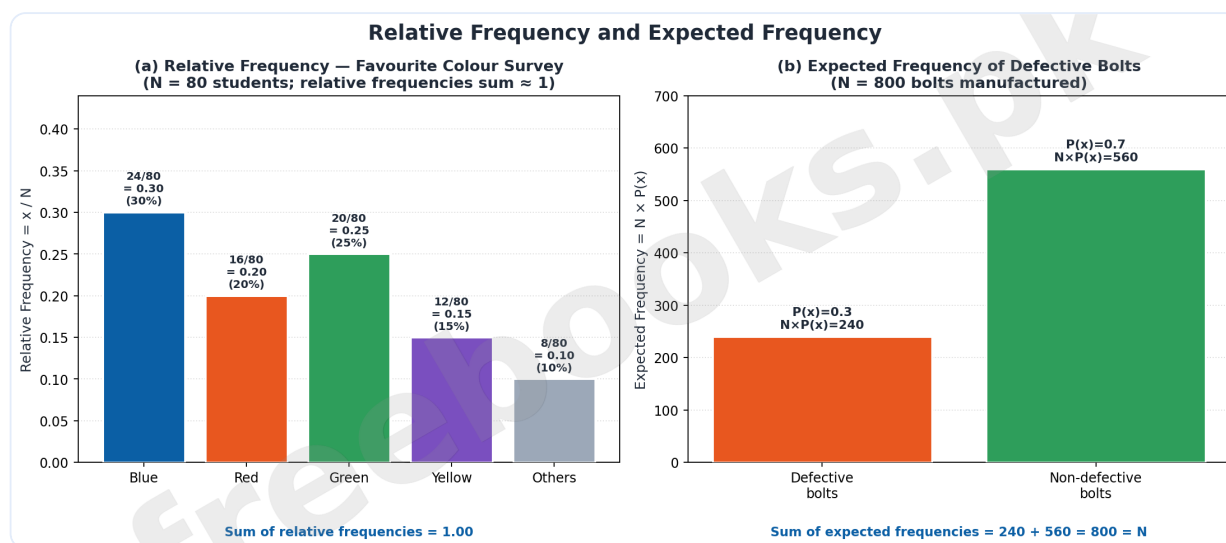
Probability of an Event and Its Complement: A two-panel figure: the left panel shows paired bars for three example events, each split into $P(A)$ and $P(A')$ segments that sum to 1; the right panel shows a 0-to-1 probability number line marking impossible, unlikely, equally likely, likely, and certain events.





Probability of an Event and Its Complement

Relative Frequency and Expected Frequency: A two-panel figure: the left panel is a bar chart of relative frequencies from an 80-student favourite-colour survey, with percentage labels summing to approximately 1; the right panel compares given probabilities $P(x)$ against expected frequencies $N \times P(x)$ for defective and non-defective bolts out of 800.



Relative Frequency and Expected Frequency



Solved Examples

Example 1: Probability of a Single Event on One Die

Problem: A fair die is rolled once. Find the probability of getting a number greater than 4.

1. Write the sample space: $S = \{1, 2, 3, 4, 5, 6\}$, so $n(S) = 6$.
2. Identify the favourable outcomes: numbers greater than 4 are $A = \{5, 6\}$, so $n(A) = 2$.
3. Apply the formula: $P(A) = n(A)/n(S) = 2/6 = 1/3$.
4. Check the range: since $0 \leq 1/3 \leq 1$, the answer is a valid probability.

Example 2: Probability Using the Sample Space of Two Dice

Problem: Two fair dice are rolled together. Find the probability that the sum of the two numbers shown is 8.

1. Write the sample space size: $n(S) = 6 \times 6 = 36$ equally likely ordered pairs.
2. List the favourable outcomes where the sum equals 8: (2,6), (3,5), (4,4), (5,3), (6,2).
3. Count the favourable outcomes: $n(A) = 5$.
4. Apply the formula: $P(A) = n(A)/n(S) = 5/36$.

Example 3: Complement of an Event — Not Getting a Six

Problem: A fair die is thrown once. Find the probability of NOT getting a 6.

1. Find the probability of the original event: $P(\text{getting a 6}) = n(A)/n(S) = 1/6$.
2. Apply the complement rule: $P(\text{not getting a 6}) = 1 - P(A) = 1 - 1/6$.
3. Simplify: $1 - 1/6 = 6/6 - 1/6 = 5/6$.
4. Verify: $P(A) + P(A') = 1/6 + 5/6 = 6/6 = 1$, confirming the complement rule holds.



Example 4: Complement for Two Dice — No Double Six

Problem: Two fair dice are thrown together. Find the probability that they do NOT both show a 6 (i.e. not a double six).

1. Write the sample space size: $n(S) = 36$.
2. Identify the double-six outcome: only (6,6), so $n(\text{double six}) = 1$, giving $P(\text{double six}) = 1/36$.
3. Apply the complement rule: $P(\text{not double six}) = 1 - P(\text{double six}) = 1 - 1/36$.
4. Simplify: $1 - 1/36 = 36/36 - 1/36 = 35/36$.

Example 5: Cards Probability with Complement

Problem: A card is drawn at random from a well-shuffled standard deck of 52 playing cards. Find the probability that it is a heart, and the probability that it is neither a spade nor a heart.

1. Note the deck structure: 52 cards in 4 suits (hearts, diamonds, clubs, spades) of 13 cards each.
2. Find $P(\text{heart})$: $n(\text{heart}) = 13$, so $P(\text{heart}) = 13/52 = 1/4$.
3. Find $P(\text{spade or heart})$: spades and hearts together give $n(A) = 13 + 13 = 26$, so $P(A) = 26/52 = 1/2$.
4. Apply the complement rule for 'neither spade nor heart': $P(\text{neither}) = 1 - P(\text{spade or heart}) = 1 - 1/2 = 1/2$.

Example 6: Relative Frequency from a Survey

Problem: In a survey of 80 students, favourite colour was recorded as: Blue 24, Red 16, Green 20, Yellow 12, Others 8. Find the relative frequency of each colour and confirm the relative frequencies sum to approximately 1.

1. Note the total: $N = \text{Sum of } f = 24 + 16 + 20 + 12 + 8 = 80$ students.



2. Compute each relative frequency as x/N : Blue = $24/80 = 0.30$, Red = $16/80 = 0.20$, Green = $20/80 = 0.25$.
3. Continue: Yellow = $12/80 = 0.15$, Others = $8/80 = 0.10$.
4. Sum the relative frequencies: $0.30 + 0.20 + 0.25 + 0.15 + 0.10 = 1.00$, confirming the sum is approximately 1.

Example 7: Expected Frequency of Getting a 1 or 6 in 300 Rolls

Problem: A fair die is rolled 300 times. Find the expected number of times a 1 or a 6 appears.

1. Find the probability of the event: $P(1 \text{ or } 6) = n(A)/n(S) = 2/6 = 1/3$.
2. Note the total number of trials: $N = 300$.
3. Apply the expected frequency formula: Expected frequency = $N \times P(A) = 300 \times (1/3)$.
4. Simplify: $300 \times (1/3) = 100$, so a 1 or a 6 is expected to appear about 100 times in 300 rolls.

Example 8: Expected Frequency of Defective Bolts

Problem: A factory finds from experience that the probability of a manufactured bolt being defective is 0.3. Out of 800 bolts produced, find the expected number of defective bolts and the expected number of non-defective bolts.

1. Note the given probabilities: $P(\text{defective}) = 0.3$, so by the complement rule $P(\text{non-defective}) = 1 - 0.3 = 0.7$.
2. Note the total number of trials: $N = 800$ bolts.
3. Expected defective bolts = $N \times P(\text{defective}) = 800 \times 0.3 = 240$.
4. Expected non-defective bolts = $N \times P(\text{non-defective}) = 800 \times 0.7 = 560$; check: $240 + 560 = 800 = N$.



Short Questions & Answers

What is a sample space? Give an example.

A sample space is the set of all possible outcomes of an experiment. For example, when a single die is rolled, the sample space is $S = \{1, 2, 3, 4, 5, 6\}$, so $n(S) = 6$.

What is the difference between an outcome and a favourable outcome?

An outcome is any single result produced by an experiment, while a favourable outcome is specifically an outcome that matches the event we are interested in. For example, when rolling a die for the event 'even number', all six numbers are outcomes, but only 2, 4, and 6 are favourable outcomes.

What is the complement of an event, and what formula is used to find its probability?

The complement of an event A , written A' , is the event that A does not occur, consisting of every outcome in the sample space not included in A . Its probability is found using the complement rule, $P(A') = 1 - P(A)$.

Why must the sum of relative frequencies in a complete distribution be approximately 1?

Relative frequency is defined as x/N for each category, where x is the frequency of that category and N is the total frequency of all categories combined. Adding every category's relative frequency together gives $(\text{Sum of } x)/N = N/N = 1$, since the sum of all the individual frequencies equals the total N .

What is expected frequency, and what formula is used to calculate it?

Expected frequency is the predicted number of times an event should occur in a given number of trials, based on its known probability. It is calculated as $\text{Expected frequency} = N \times P(A)$, where N is the total number of trials and $P(A)$ is the probability of the event.

Give one example each of a certain event and an impossible event.

A certain event is one that is sure to happen, such as getting a number between 1 and 6 when rolling an ordinary die, which has probability $P(A) = 1$. An impossible event can never happen, such as getting a 7 on an ordinary six-sided die, which has probability $P(A) = 0$.



How does the sample space of two dice rolled together differ from that of a single die?

A single die has a sample space of $S = \{1, 2, 3, 4, 5, 6\}$, so $n(S) = 6$. When two dice are rolled together, every outcome is an ordered pair of results, giving a sample space of $n(S) = 6 \times 6 = 36$ equally likely outcomes.

What is the difference between theoretical probability and relative frequency?

Theoretical probability is calculated in advance using $P(A) = n(A)/n(S)$, assuming every outcome in the sample space is equally likely, such as with a fair die or coin. Relative frequency, x/N , is instead calculated from actually recorded data after an experiment or survey has been carried out, and is used as a practical estimate of probability when outcomes are not known to be equally likely in advance.

Long Questions & Answers

Explain the basic concepts of probability and how the probability of a single event and its complement are calculated.

What is an experiment, sample space, and event in probability?

An experiment is any process that generates a result, such as tossing a coin or rolling a die. The sample space S is the set of all possible outcomes of that experiment — for a single die, $S = \{1, 2, 3, 4, 5, 6\}$, so $n(S) = 6$ — and each individual outcome within it is called a sample point. An event is any subset of the sample space representing a particular condition of interest, such as $A = \{2, 4, 6\}$ for 'an even number appears' on a die, giving $n(A) = 3$.

What is the formula for the probability of an event, and what are its lower and upper limits?

The probability of an event A is given by $P(A) = n(A)/n(S)$, where $n(A)$ is the number of favourable outcomes and $n(S)$ is the total number of possible outcomes in the sample space. This ratio always lies between 0 and 1, written $0 \leq P(A) \leq 1$, because the number of favourable outcomes can never be negative



and can never exceed the total number of outcomes. A probability closer to 0 indicates the event is unlikely, while a probability closer to 1 indicates it is likely.

What are certain, impossible, likely, unlikely, and equally likely events?

A certain event is one that is sure to occur every time the experiment is performed, with $P(A) = 1$, such as getting a number from 1 to 6 on an ordinary die. An impossible event can never occur, with $P(A) = 0$, such as getting a 7 on that same die. Between these extremes, an event with probability greater than 0.5 is called likely, one with probability less than 0.5 is called unlikely, and one with probability exactly 0.5 is called equally likely, such as getting heads on a single fair coin toss.

How is the complement of an event calculated, and what real-life problems use it?

The complement of an event A , written A' , is the event that A does not happen, and it is calculated using the complement rule $P(A') = 1 - P(A)$, since $P(A) + P(A')$ always equals 1. This is useful in real life whenever the 'not occurring' case is easier to work with than listing every favourable outcome directly — for example, finding the probability that a missile misses a target given its probability of hitting it, or finding the probability of drawing neither a spade nor a heart from a deck of cards by subtracting $P(\text{spade or heart})$ from 1.

Explain relative frequency and expected frequency as estimates of probability, including their real-life applications.

What is relative frequency, and how does it differ from theoretical probability?

Relative frequency is an estimate of probability calculated from actual recorded data, defined as $\text{Relative frequency} = x/N$, where x is how many times a specific event occurred and N is the total number of observations. It differs from theoretical probability, $P(A) = n(A)/n(S)$, which is calculated in advance assuming every outcome in a sample space is equally likely (as with a fair die or coin); relative frequency instead reflects what was actually observed in a survey, experiment, or set of records, and does not require the outcomes to be equally likely beforehand.



How is relative frequency calculated from a frequency table, and what should the relative frequencies sum to?

To calculate relative frequency from a frequency table, first find the total frequency N by adding up the frequencies of every category ($N = \text{Sum of } f$), then divide each individual category's frequency x by this total to get x/N for that category. When every category's relative frequency is added together, the sum should always come out to approximately 1 (exactly 1 without rounding), since the individual frequencies themselves add up to the total N .

What is expected frequency, and how is it calculated?

Expected frequency is the number of times an event is predicted to occur in a given number of trials, based on its known or assumed probability. It is calculated using $\text{Expected frequency} = N \times P(A)$, where N is the total number of trials and $P(A)$ is the probability of the event; just like relative frequencies sum to approximately 1, the expected frequencies of every category in a distribution sum to approximately N , the fixed total number of trials.

What real-life applications use relative frequency and expected frequency?

Relative frequency is commonly used to summarise survey results, such as finding what proportion of students prefer each of several favourite colours out of a sample of 80 students. Expected frequency is used to turn a known probability into a practical prediction, such as calculating that a fair die rolled 300 times should show a 1 or a 6 about 100 times, or that a factory producing 800 bolts with a 0.3 probability of a defect should expect about 240 defective and 560 non-defective bolts, information used for planning quality control and inspection.

Multiple Choice Questions (MCQs)

Each individual member of a sample space is called a:

- A. Event
- B. Sample point
- C. Favourable outcome
- D. Frequency

Answer: B. Sample point



Explanation: Each individual outcome that belongs to a sample space is called a sample point.

An outcome that matches the particular result we are interested in is called a:

- A. Sample space
- B. Complement
- C. Favourable outcome
- D. Experiment

Answer: C. Favourable outcome

Explanation: A favourable outcome is an outcome of an experiment that satisfies the event under consideration.

Relative frequency is defined as:

- A. $n(A)/n(S)$
- B. $N \times P(A)$
- C. x/N (frequency of the event / total frequency)
- D. $1 - P(A)$

Answer: C. x/N (frequency of the event / total frequency)

Explanation: Relative frequency is calculated from recorded data as the frequency of the specific event divided by the total frequency, x/N .

Expected frequency is also known as:

- A. Relative frequency
- B. Theoretical frequency
- C. Class frequency
- D. Complement frequency

Answer: B. Theoretical frequency

Explanation: Expected frequency, calculated as $N \times P(A)$, is also called theoretical frequency because it is the number of occurrences predicted by probability theory.

The sum of the expected frequencies of all categories in a distribution is approximately equal to:

- A. 1
- B. 0.5
- C. The fixed total number of trials, N
- D. The probability of a single event



Answer: C. The fixed total number of trials, N

Explanation: Since expected frequency = $N \times P(A)$ for each category, and the probabilities sum to 1, the expected frequencies sum to approximately N , the total number of trials.

The probability of an event A is defined as:

- A. $P(A) = n(S)/n(A)$
- B. $P(A) = n(A) + n(S)$
- C. $P(A) = n(A)/n(S)$
- D. $P(A) = n(A) \times n(S)$

Answer: C. $P(A) = n(A)/n(S)$

Explanation: Probability is defined as the number of favourable outcomes $n(A)$ divided by the total number of possible outcomes $n(S)$.

An event whose probability is greater than 0.5 is called a:

- A. Certain event
- B. Impossible event
- C. Likely event
- D. Equally likely event

Answer: C. Likely event

Explanation: An event with a probability greater than 0.5 is described as a likely event, since it is more likely to occur than not.

If n fair dice are rolled together, the size of the sample space is:

- A. $6n$
- B. n^6
- C. 6^n
- D. $6 + n$

Answer: C. 6^n

Explanation: Each die has 6 equally likely outcomes, so for n dice rolled together the sample space size is 6^n , e.g. $6^2 = 36$ for two dice.

Two fair dice are rolled together. The probability that both dice show a 2 is:

- A. $1/6$
- B. $1/12$
- C. $1/36$
- D. $2/36$



Answer: C. $1/36$

Explanation: Only one outcome out of 36, namely (2,2), gives a double two, so the probability is $1/36$.

A card is drawn from a standard 52-card deck. The probability that it is NOT a jack and NOT a king is:

A. $8/52$

B. $44/52$

C. $4/52$

D. $48/52$

Answer: B. $44/52$

Explanation: There are 4 jacks and 4 kings, 8 cards in total to exclude, leaving $52 - 8 = 44$ favourable cards, so the probability is $44/52$.

Quick Revision Summary

- Probability $P(A) = n(A)/n(S)$, where $n(A)$ = favourable outcomes and $n(S)$ = total possible outcomes in the sample space.
- Every probability satisfies $0 \leq P(A) \leq 1$; $P(A) = 0$ is an impossible event, $P(A) = 1$ is a certain event.
- A sample space S is the set of all possible outcomes; each outcome in S is a sample point; an event is any subset of S .
- For a single die $n(S) = 6$; for two dice rolled together $n(S) = 6 \times 6 = 36$; in general, for n dice $n(S) = 6^n$.
- The complement rule states $P(A') = 1 - P(A)$, and $P(A) + P(A') = 1$, since every outcome is either in A or in A' but not both.
- A standard deck has 52 cards, 4 suits of 13 cards each, so $P(\text{any single suit}) = 13/52 = 1/4$.
- Relative frequency $= x/N$ estimates probability from recorded data; the relative frequencies of a complete distribution sum to approximately 1.
- Expected frequency $= N \times P(A)$ predicts how often an event should occur in N trials; expected frequencies of a distribution sum to approximately N .
- Likely events have $P(A) > 0.5$, unlikely events have $P(A) < 0.5$, and equally likely events have $P(A) = 0.5$.
- Probability, relative frequency, and expected frequency are used in real life for target/risk problems, surveys, and quality-control forecasting.



Exam Tips

- Always write out the sample space S (or at least count $n(S)$) before trying to count favourable outcomes — it prevents careless mistakes.
- For two-dice problems, sketching the 6×6 grid of outcomes makes it much easier to count favourable outcomes accurately than trying to list them from memory.
- When an event is described as 'not' something, check whether the complement rule $P(A') = 1 - P(A)$ is faster than counting favourable outcomes directly.
- Always simplify your final probability fraction, and sanity-check that it lies between 0 and 1.
- When using relative frequency, add up all the relative frequencies at the end — they should total approximately 1; if not, recheck your arithmetic.
- For expected frequency problems, always check that the expected frequencies of every category sum back to the given total number of trials, N .

