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FSc/ICS Part-II (2nd Year) — PECTAA Board

UNIT 1

Graphical Representation of Functions

Mathematics • Class 12 • 2nd Year • PECTAA Board • Free PDF Notes

Unit 1: Graphical Representation of Functions

This unit builds a graphical toolkit for functions: sketching a quadratic from its factored form and predicting a quadratic equation from a given graph, classifying functions as algebraic or transcendental, and studying the key transcendental function families -- exponential, hyperbolic, and logarithmic -- through their domains, ranges, intercepts, monotonicity, and asymptotes.

It then connects these ideas visually: the graph of an inverse function is the reflection of the original graph about the line y=x (which is why y=ln x mirrors y=e^x), and any graph can be moved or resized using horizontal/vertical shifts and horizontal/vertical scaling. The unit closes with solving exponential and logarithmic equations and inequalities, and applying these functions to real-world growth and decay, sound intensity (decibels), and compound interest problems.

Learning Objectives

- Sketch the graph of a quadratic function given in factored form, and predict a quadratic function from its graph
- Classify functions as algebraic or transcendental, and as fundamental or non-fundamental transcendental functions
- State and apply the domain, range, intercepts, monotonicity, and asymptotic behaviour of exponential functions and sketch their graphs
- Define the hyperbolic and inverse hyperbolic functions and derive their logarithmic forms
- State and apply the laws of logarithms and sketch the graph of a logarithmic function as the inverse of an exponential function
- Construct the graph of an inverse function by reflecting the original graph about the line y=x
- Apply horizontal/vertical shifts and horizontal/vertical scaling to transform the graph of a function
- Solve exponential and logarithmic equations and inequalities, and apply these functions to growth/decay, sound intensity, and compound interest problems

Key Concepts

1.1 Sketching and Predicting Quadratic Graphs Using Factors

A quadratic y=a(x-h)(x-k) (h<=k) can be sketched directly from its factors: the graph cuts the x-axis at x=h and x=k, its vertex lies at x=(h+k)/2, its end behaviour (y -> +infinity or y -> -infinity as x -> +-infinity) is fixed by the sign of a, and its y-intercept is found by setting x=0 (giving y=ahk). When h=k, the factor (x-h)^2 makes the parabola touch rather than cross the x-axis, with vertex (h,0).

Conversely, if the x-intercepts x=h and x=k of a quadratic graph are known, the function can be written as f(x)=a(x-h)(x-k), and substituting the coordinates of any third point on the curve determines the remaining constant a -- letting a full equation be recovered directly from a picture of its graph.

1.2 Algebraic and Transcendental Functions

A function y=f(x) is algebraic if it satisfies a polynomial equation in y with polynomial coefficients in x -- equivalently, if it is built from x using finitely many algebraic operations (addition, subtraction, multiplication, division, and nth roots). A function that is not algebraic is called transcendental.

Transcendental functions split further into fundamental transcendental functions -- the standard building blocks such as y=a^x, y=ln x, y=sin x, y=sin^-1 x -- and non-fundamental transcendental functions, formed by combining these with algebraic functions or with each other through composition (e.g. y=sin x + ln x, y=e^(x^2)).

1.3 Exponential, Hyperbolic, and Inverse Hyperbolic Functions

The exponential function f(x)=a^x (a>0, a<>1) has domain]-infinity,infinity[, range]0,infinity[, is continuous and always positive, cuts the y-axis at (0,1), has no x-intercept, and has horizontal asymptote y=0. It is strictly increasing when a>1 and strictly decreasing when 0<a<1, and it is injective in both cases, so every horizontal line meets its graph at most once.

Hyperbolic functions sinh x=(e^x-e^-x)/2, cosh x=(e^x+e^-x)/2, and tanh x=sinh x/cosh x (plus their reciprocals coth, sech, csch) are built from e^x and behave like trigonometric functions but arise from the hyperbola x^2-y^2=1 rather than the circle. Their inverses -- such as sinh^-1 x=ln(x+sqrt(x^2+1)) and cosh^-1 x=ln(x+sqrt(x^2-1)) on [1,infinity) -- are derived by solving the defining exponential equation for x algebraically using the quadratic formula.

1.4 Logarithmic Functions and Their Graphs

The logarithmic function y=log_a x is defined as the inverse of the exponential function y=a^x, with domain]0,infinity[and range]-infinity,infinity[. It is injective, has vertical asymptote x=0, cuts the x-axis at (1,0), has no y-intercept, and is strictly increasing when a>1 and strictly decreasing when 0<a<1. The product, quotient, and power laws of logarithms, together with the change-of-base formula log_a x=ln x/ln a, are the standard tools for simplifying logarithmic expressions and equations.

The modulus function f(x)=|x| produces a V-shaped graph made of the two lines y=x (for x>=0) and y=-x (for x<0), meeting at the origin.

1.5 Graphs of Inverse Functions and Transformations

A function has an inverse exactly when it is bijective (one-to-one and onto); its inverse graph is the reflection of the original graph about the line y=x, since interchanging the roles of input and output corresponds exactly to reflecting across the line where input equals output. This is why the graph of y=ln x is the mirror image of y=e^x about y=x.

A graph can be transformed by shifting -- y=f(x-h)+k moves the graph h units horizontally (right if h>0, left if h<0) and k units vertically (up if k>0, down if k<0) -- or by scaling -- y=a f(bx) stretches/compresses the graph vertically by a and horizontally by 1/b (for a,b>0), where a>1 stretches vertically and 0<a<1 compresses vertically, while b>1 compresses horizontally and 0<b<1 stretches horizontally.

1.6 Exponential/Logarithmic Equations, Inequalities, and Real-World Applications

Exponential and logarithmic equations are solved by matching bases (using injectivity of a^x or log_a x) or by taking logarithms of both sides; the domain must always be checked first, since logarithmic expressions require positive arguments and positive, non-unit bases. The same injectivity/monotonicity facts convert exponential and logarithmic inequalities into ordinary algebraic inequalities, again subject to the domain restrictions of the original expressions.

These functions model real growth and decay via N=N0 e^(kt) (k>0 for growth, k<0 for decay), sound intensity in decibels via L=10 log(I/I0) with threshold I0=10^-12 W/m^2, and compound interest via the discrete formula A=P(1+r/n)^(nt) or the continuous formula A=Pe^(rt) -- each a direct application of solving an exponential or logarithmic equation for the unknown quantity.

Important Definitions

What is an algebraic function?

A function y=f(x) obtained from x using a finite number of algebraic operations (addition, subtraction, multiplication, division, and nth roots), i.e. one satisfying a polynomial equation in y with polynomial coefficients in x.

What is a transcendental function?

A function that is not algebraic, such as y=e^x, y=ln x, y=sin x, or y=sin^-1 x.

What is the natural exponential function?

f(x)=e^x, where e is the number defined by the infinite series sum(1/n!) for n=0 to infinity, with e is approximately 2.71828.

What is a horizontal asymptote?

A horizontal line y=c that a function's graph approaches (but need not touch) as x -> +infinity or x -> -infinity.

What is a vertical asymptote?

A vertical line x=c such that f(x) -> +-infinity as x -> c from the left or right.

What are the hyperbolic functions sinh x and cosh x?

sinh x = (e^x - e^-x)/2 and cosh x = (e^x + e^-x)/2; they satisfy cosh^2 x - sinh^2 x = 1, the hyperbola analogue of sin^2+cos^2=1.

What is a logarithmic function?

y=log_a x (a>0, a<>1, x>0), defined as the inverse of the exponential function y=a^x, i.e. a^y=x.

How is the graph of an inverse function obtained from the original graph?

By reflecting the original graph about the line y=x.

What does the general shift formula y=f(x-h)+k represent?

A horizontal shift of h units (right if h>0, left if h<0) and a vertical shift of k units (up if k>0, down if k<0).

What is the continuous compound interest formula?

A = P e^(rt), where P is the principal, r is the annual interest rate (decimal), and t is time.

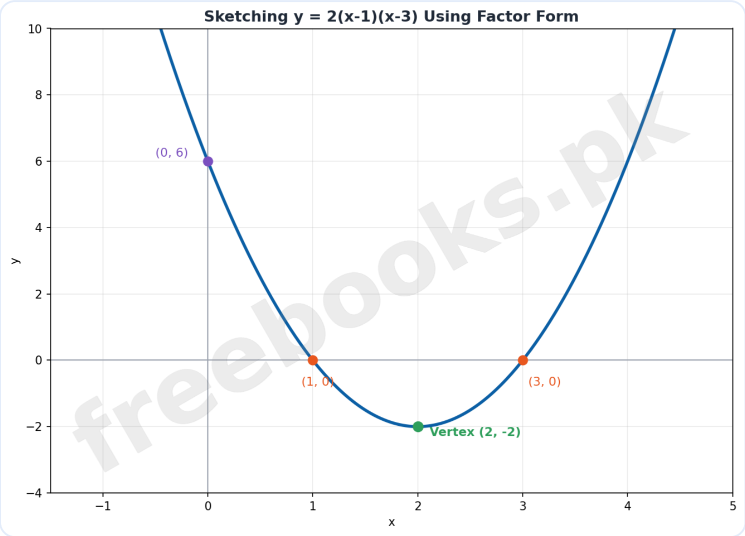
Key Facts and Relations

Topic	Key Fact / Relation
Quadratic graph via factors	y = a(x-h)(x-k); x-intercepts x=h,k; vertex at x=(h+k)/2; y-intercept y=ahk
Exponential function	y = a^x (a>0, a<>1); domain]-inf,inf[, range]0,inf[, y-intercept (0,1), asymptote y=0
Hyperbolic functions	sinh x=(e^x-e^-x)/2, cosh x=(e^x+e^-x)/2, tanh x=sinh x/cosh x; cosh^2 x - sinh^2 x = 1
Inverse hyperbolic sine/cosine	sinh^-1 x = ln(x+sqrt(x^2+1)); cosh^-1 x = ln(x+sqrt(x^2-1)), x>=1
Laws of logarithms	log_a(xy)=log_a x+log_a y; log_a(x/y)=log_a x-log_a y; log_a(x^t)=t log_a x; log_a x = ln x/ln a
Logarithmic function	y = log_a x (a>0, a<>1, x>0); domain]0,inf[, range]-inf,inf[, x-intercept (1,0), asymptote x=0
General shift of a graph	y = f(x-h) + k (h: horizontal shift, k: vertical shift)

Topic	Key Fact / Relation
General scaling of a graph	y = a f(bx), a>0, b>0 (a: vertical scale factor, 1/b: horizontal scale factor)
Growth/decay model	N = N0 e^(kt) (k>0 growth, k<0 decay)
Sound level (decibels) / Compound interest	L = 10 log(I/I0), I0=10^-12 W/m^2; A = P(1+r/n)^(nt) (discrete); A = Pe^(rt) (continuous)

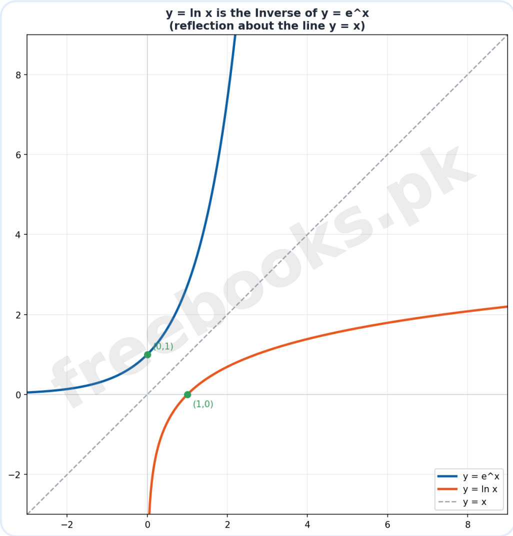
Diagrams

Sketching y = 2(x-1)(x-3) Using Factor Form: A parabola showing the x-intercepts at x=1 and x=3, the vertex at (2,-2), and the y-intercept at (0,6), illustrating the factor-form sketching method



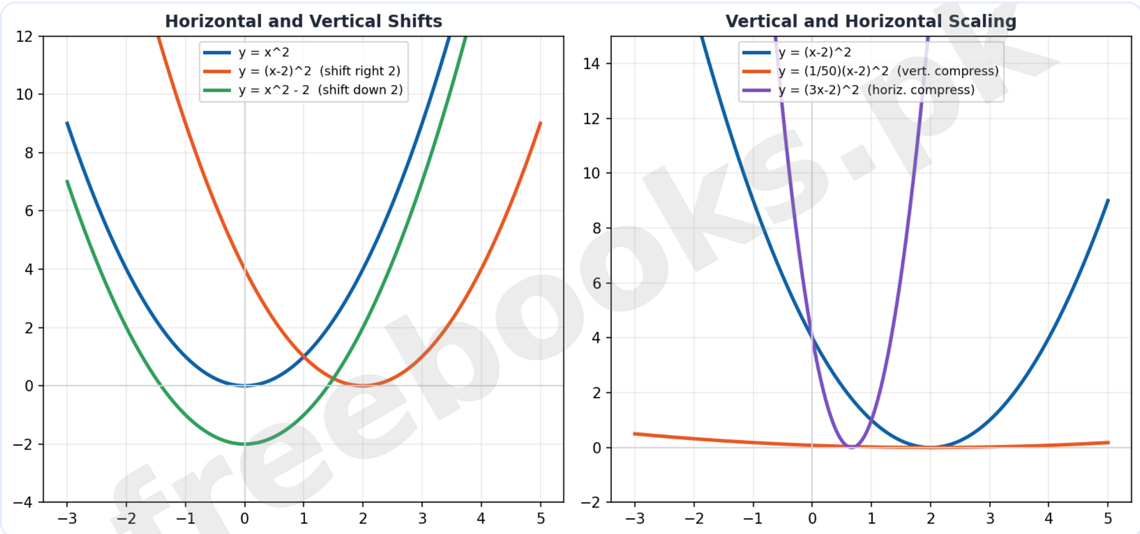
Sketching y = 2(x-1)(x-3) Using Factor Form

y = ln x is the Inverse of y = e^x: The graphs of y=e^x and y=ln x plotted together with the line y=x, showing that the two curves are reflections of each other about y=x



y = ln x is the Inverse of y = e^x

Horizontal/Vertical Shifts and Scaling of Graphs: Two panels showing y=x^2 transformed by horizontal and vertical shifts (left) and by vertical and horizontal scaling (right)



Horizontal/Vertical Shifts and Scaling of Graphs

Solved Examples

Example 1: Sketching a Quadratic Graph Using Factors

Problem: Sketch the graph of y = 2(x-1)(x-3).

1. Find the x-intercepts by setting each factor to zero: x-1=0 and x-3=0, giving x=1 and x=3, so the graph meets the x-axis at (1,0) and (3,0).

2. Find the vertex by evaluating y at $x=(1+3)/2=2$: $y=2(2-1)(2-3)=-2$, so the vertex is $(2,-2)$.
3. Determine end behaviour: since $a=2>0$, $y \to +\infty$ as $x \to \pm\infty$.
4. Find the y-intercept by setting $x=0$: $y=2(-1)(-3)=6$, so the y-intercept is $(0,6)$.
5. Using the intercepts $(1,0)$, $(3,0)$, $(0,6)$ and vertex $(2,-2)$, sketch the upward-opening parabola.

Example 2: Predicting a Quadratic Function from Its Graph

Problem: Find an equation of the graph whose x-intercepts are $x=1$ and $x=2$ and which passes through the point $(3,4)$.

1. Since the x-intercepts are $x=1$ and $x=2$, write the function in factored form: $y=a(x-1)(x-2)$.
2. Substitute the known point $(3,4)$: $4=a(3-1)(3-2)=2a$.
3. Solve for a: $a=4/2=2$.
4. Substitute $a=2$ back into the factored form: $y=2(x-1)(x-2)=2(x^2-3x+2)$.
5. Final answer: $y=2x^2-6x+4$.

Example 3: Proving a Hyperbolic Identity

Problem: Prove that $\cosh^2 x - \sinh^2 x = 1$.

1. Write both functions in terms of e^x: $\cosh x=(e^x+e^{-x})/2$ and $\sinh x=(e^x-e^{-x})/2$.
2. Square each: $\cosh^2 x = (e^{2x}+e^{-2x}+2)/4$ and $\sinh^2 x = (e^{2x}+e^{-2x}-2)/4$.
3. Subtract: $\cosh^2 x - \sinh^2 x = [(e^{2x}+e^{-2x}+2)-(e^{2x}+e^{-2x}-2)]/4$.
4. Simplify the numerator: $(e^{2x}+e^{-2x}+2-e^{2x}-e^{-2x}+2)/4 = 4/4 = 1$.
5. Final answer: $\cosh^2 x - \sinh^2 x = 1$.

Example 4: Deriving the Logarithmic Form of sinh^-1 x

Problem: Prove that sinh^-1 x = ln(x + sqrt(x^2+1)).

1. Let y=sinh^-1 x, so x=sinh y = (e^y-e^-y)/2, giving 2x=e^y-1/e^y.
2. Multiply through by e^y: 2x e^y = e^(2y) - 1, i.e. (e^y)^2 - 2x e^y - 1 = 0, a quadratic in e^y.
3. Apply the quadratic formula: e^y = [2x +- sqrt(4x^2+4)]/2 = x +- sqrt(x^2+1).
4. Since x - sqrt(x^2+1) < 0 for all real x but e^y must be positive, reject the negative sign: e^y = x + sqrt(x^2+1).
5. Take the natural log of both sides: y = ln(x+sqrt(x^2+1)). Final answer: sinh^-1 x = ln(x+sqrt(x^2+1)).

Example 5: Solving an Exponential Equation by Matching Bases

Problem: Solve the equation 4^(x-1) = 8.

1. Note the domain: 4^(x-1) is defined for all real x, so the domain is]-infinity,infinity[.
2. Write both sides with base 2: 4=2^2 and 8=2^3, so (2^2)^(x-1)=2^3, i.e. 2^(2x-2)=2^3.
3. Since the exponential function is one-to-one (injective), the exponents must be equal: 2x-2=3.
4. Solve for x: 2x=5, so x=5/2.
5. Final answer: solution set = {5/2}.

Example 6: Solving a Logarithmic Inequality

Problem: Solve the inequality log(2x-1) >= log(5-3x).

1. Find the domain: 2x-1>0 gives x>1/2, and 5-3x>0 gives x<5/3, so the domain is 1/2 < x < 5/3.

2. Since the logarithmic function with base 10 is strictly increasing, the inequality is equivalent to comparing the arguments directly: 2x-1 >= 5-3x.
3. Solve the resulting linear inequality: 2x+3x >= 5+1, so 5x >= 6, giving x >= 6/5.
4. Combine with the domain restriction 1/2 < x < 5/3: this gives 6/5 <= x < 5/3.
5. Final answer: solution set = [6/5, 5/3).

Example 7: Applying Multiple Graph Transformations in Order

Problem: Let f(x) = x/(x+1). Find the function after: (i) vertical stretch by 2, (ii) horizontal stretch by 3, (iii) shift 2 units right, (iv) shift 1 unit down.

1. (i) Vertical stretch by 2: y=2f(x)=2x/(x+1). Call this g(x)=2x/(x+1).
2. (ii) Horizontal stretch by 3 means y=g(x/3): substituting gives y=2(x/3)/((x/3)+1)=2x/(x+3). Call this s(x)=2x/(x+3).
3. (iii) Shift 2 units right means y=s(x-2)=2(x-2)/((x-2)+3)=(2x-4)/(x+1). Call this t(x)=(2x-4)/(x+1).
4. (iv) Shift 1 unit down means y=t(x)-1=(2x-4)/(x+1) - 1 = (2x-4-(x+1))/(x+1) = (x-5)/(x+1).
5. Final answer: y = (x-5)/(x+1).

Example 8: Real-World Application: Continuous Compound Interest

Problem: A sum of Rs. 1000 is invested at a rate of 5% per year compounded continuously for 3 years. Find the final amount.

1. Identify the given values: P=1000, r=5%=0.05, t=3.
2. Apply the continuous compounding formula: A = P e^(rt).
3. Substitute the values: A = 1000 e^(0.05x3) = 1000 e^0.15.
4. Evaluate: e^0.15 is approximately 1.1618, so A is approximately 1000 x 1.1618 = 1161.8.

> 5. Final answer: the final amount after 3 years is approximately Rs. 1161.8.

Short Questions & Answers

What determines whether an exponential function $y=a^x$ is increasing or decreasing?

The base a: strictly increasing if $a>1$, strictly decreasing if $0<a<1$.

What is the horizontal asymptote of every exponential function $y=a^x$?

$y=0$, the x-axis.

What is the vertical asymptote of every logarithmic function $y=\log_a x$?

$x=0$, the y-axis.

Through which point does every exponential function $y=a^x$ pass?

(0,1), since $a^0=1$.

Through which point does every logarithmic function $y=\log_a x$ pass?

(1,0), since $\log_a 1=0$.

How is the graph of an inverse function related to the original graph?

It is the reflection of the original graph about the line $y=x$.

What is the shape of the graph of the modulus function $f(x)=|x|$?

A V-shape, made of the lines $y=x$ for $x>=0$ and $y=-x$ for $x<0$, meeting at the origin.

Long Questions & Answers

Explain how to sketch and predict the graph of a quadratic function using its factored form, with the roles of intercepts and vertex.

How do the x-intercepts of $y=a(x-h)(x-k)$ relate to its factors?

Setting each factor to zero gives $x=h$ and $x=k$ directly, so the graph crosses the x-axis exactly at these two points (or touches it once at $x=h$ if $h=k$).

How is the vertex of the parabola located?

The vertex lies midway between the two x-intercepts, at x=(h+k)/2; substituting this x-value back into the function gives the vertex's y-coordinate.

How does the sign of a determine the graph's end behaviour?

If a>0, y -> +infinity as x -> +-infinity (the parabola opens upward); if a<0, y -> -infinity as x -> +-infinity (it opens downward).

How can a quadratic function be predicted from its graph?

If the x-intercepts h and k are known, write f(x)=a(x-h)(x-k), then substitute the coordinates of any third point on the curve to solve for the remaining constant a.

Compare exponential and logarithmic functions: their domains, ranges, monotonic behaviour, and how their graphs relate to each other.

What are the domain and range of y=a^x versus y=log_a x?

The exponential function y=a^x has domain]-infinity,infinity[and range]0,infinity[; the logarithmic function y=log_a x has domain]0,infinity[and range]-infinity,infinity[-- domain and range are exactly swapped, since the two are inverses.

How does the base a affect monotonic behaviour in each case?

For both y=a^x and y=log_a x, the function is strictly increasing when a>1 and strictly decreasing when 0<a<1 -- the same base condition governs monotonicity in both families.

What asymptotes do these two graphs have?

y=a^x has horizontal asymptote y=0 (the x-axis); y=log_a x has vertical asymptote x=0 (the y-axis) -- again mirror images of each other.

Why is the graph of y=log_a x the reflection of y=a^x about y=x?

Because y=log_a x is defined as the inverse of y=a^x, and the graph of any inverse function is obtained by reflecting the original graph about the line y=x, where input and output roles are interchanged.

Multiple Choice Questions (MCQs)

The graph of y=a(x-h)(x-k) crosses the x-axis at:

A. x=a only

B. x=h and x=k

C. x=(h+k)/2 only

D. It never crosses the x-axis

Answer: B. x=h and x=k

Explanation: Setting each factor to zero gives the x-intercepts x=h and x=k directly.

A function is called transcendental if it is:

A. Always positive

B. Not algebraic

C. A polynomial

D. Always increasing

Answer: B. Not algebraic

Explanation: A transcendental function is, by definition, any function that is not algebraic.

The exponential function y=a^x (a>1) is:

A. Strictly decreasing

B. Strictly increasing

C. Constant

D. Undefined for x<0

Answer: B. Strictly increasing

Explanation: When a>1, y=a^x is strictly increasing on its whole domain.

The horizontal asymptote of every exponential function y=a^x is:

A. x=0

B. y=1

C. y=0

D. y=a

Answer: C. y=0

Explanation: As x -> -infinity (for a>1) or x -> +infinity (for 0<a<1), a^x -> 0, so y=0 is the horizontal asymptote.

cosh^2 x - sinh^2 x equals:

A. 0

B. 1

C. 2x

D. e^x

Answer: B. 1

Explanation: This is the hyperbolic identity analogous to sin^2+cos^2=1, proved directly from the exponential definitions of sinh and cosh.

The logarithmic function y=log_a x has vertical asymptote:

A. x=0

B. y=0

C. x=1

D. y=1

Answer: A. x=0

Explanation: As x -> 0+, log_a x -> +-infinity depending on the base, making x=0 the vertical asymptote.

The graph of an inverse function is obtained from the original graph by:

A. Reflecting about the x-axis

B. Reflecting about the line y=x

C. Rotating 90 degrees

D. Shifting up by 1 unit

Answer: B. Reflecting about the line y=x

Explanation: Interchanging input and output corresponds geometrically to reflecting the graph about the line y=x.

In y=f(x-h)+k, a positive value of h shifts the graph:

A. Left

B. Right

C. Up

D. Down

Answer: B. Right

Explanation: A positive h shifts the graph h units to the right; the sign is opposite to what might be expected because it replaces x with x-h.

In y=af(bx) with a>1 and b>1, the graph is:

A. Compressed vertically and stretched horizontally

B. Stretched vertically and compressed horizontally

C. Unchanged

D. Reflected about the origin

Answer: B. Stretched vertically and compressed horizontally

Explanation: a>1 stretches the graph vertically by factor a, while b>1 compresses it horizontally by factor b.

The continuous compound interest formula is:

A. A=P(1+r)^t

B. A=Prt

C. A=Pe^(rt)

D. A=P+rt

Answer: C. A=Pe^(rt)

Explanation: Continuous compounding is modeled by A=Pe^(rt), the limiting case of discrete compounding as the number of periods grows without bound.

Quick Revision Summary

- y=a(x-h)(x-k): x-intercepts x=h,k; vertex at x=(h+k)/2; y-intercept y=ahk; end behaviour set by sign of a
- Algebraic functions use only +,-,x,/,and nth roots of x; transcendental functions are everything else
- Fundamental transcendental functions: a^x, ln x, sin x, sin^-1 x; combinations of these are non-fundamental
- Exponential y=a^x: domain]-inf,inf[, range]0,inf[, y-intercept (0,1), asymptote y=0, increasing if a>1, decreasing if 0<a<1
- sinh x=(e^x-e^-x)/2, cosh x=(e^x+e^-x)/2; cosh^2 x - sinh^2 x = 1; sinh^-1 x = ln(x+sqrt(x^2+1))
- Logarithmic y=log_a x: domain]0,inf[, range]-inf,inf[, x-intercept (1,0), asymptote x=0, is the inverse of y=a^x
- Laws of logarithms: log_a(xy)=log_a x+log_a y; log_a(x/y)=log_a x-log_a y; log_a(x^t)=t log_a x

- The graph of an inverse function is the reflection of the original graph about the line y=x
- Shift: y=f(x-h)+k moves h units horizontally, k units vertically; Scale: y=af(bx) scales vertically by a, horizontally by 1/b
- Exponential/logarithmic equations: match bases using injectivity, or take logs of both sides; always check the domain first
- Growth/decay: N=N0 e^(kt); Sound level: L=10 log(I/I0); Compound interest: A=P(1+r/n)^(nt) (discrete) or A=Pe^(rt) (continuous)

Exam Tips

- When sketching a quadratic from factors, always find the x-intercepts, vertex, and y-intercept in that order -- three points are enough to sketch an accurate parabola
- To classify a function as algebraic or transcendental, check whether it can be built using only +,-,x,/,and roots of x -- any exponential, logarithmic, or trigonometric piece makes it transcendental
- Memorize the four exponential/logarithmic facts as a pair: exponential passes through (0,1) with asymptote y=0; logarithmic passes through (1,0) with asymptote x=0
- For hyperbolic identity proofs, always rewrite sinh x and cosh x in terms of e^x and e^-x first, then simplify algebraically
- Before solving any exponential or logarithmic equation/inequality, find the domain first -- many 'solutions' get rejected at the end for violating the original domain
- When applying multiple graph transformations in sequence, apply them strictly in the given order, one function substitution at a time -- reordering changes the final answer