Class 10 (Science Group) · Punjab Board · English Medium

Mathematics

Complete Notes — All 13 Units

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Unit 1: Quadratic Equations

A quadratic equation is a second-degree equation in one variable -- the highest power of the unknown is 2, never higher. Quadratic equations are the gateway from linear algebra to the wider study of polynomial equations, and this unit builds the three core solution techniques (factorization, completing the square, and the quadratic formula) before extending them to more complicated equation types that LOOK unlike a quadratic on the surface but can be reduced to one by a clever substitution.

This unit also introduces radical equations -- equations where the variable sits under a square-root sign -- and the important idea of an extraneous root: a value that satisfies the squared/simplified version of an equation but fails to satisfy the original equation, and must therefore be rejected after checking.

Learning Objectives

- Define a quadratic equation and write it in standard form
- Solve a quadratic equation by factorization
- Solve a quadratic equation by completing the square
- Derive the quadratic formula using the method of completing the square
- Solve a quadratic equation using the quadratic formula
- Solve equations of the type ax^4 + bx^2 + c = 0 by reducing them to quadratic form
- Solve equations of the type ap(x) + b/p(x) = c by substitution
- Solve reciprocal equations of the type a(x^2 + 1/x^2) + b(x + 1/x) + c = 0
- Solve exponential equations where the variable occurs in the exponent
- Solve equations of the type (x+a)(x+b)(x+c)(x+d) = k where a+b = c+d
- Solve radical equations and check for extraneous roots

Key Concepts

1.1 Quadratic Equation

An equation that contains the square of the unknown quantity, but no higher power, is called a quadratic equation, or an equation of the second degree. Its standard (general) form is $ax^2 + bx + c = 0$, where a is not equal to 0 and a, b, c are real numbers -- here a is the coefficient of x^2, b is the coefficient of x, and c is the constant term. For example, $x^2 - 7x + 6 = 0$ is already in standard form, while $3x^2 + 4x = 5$ is not (it must first be rearranged to $3x^2 + 4x - 5 = 0$).

If $b = 0$ in $ax^2 + bx + c = 0$, the equation is called a pure quadratic equation -- for example $x^2 - 16 = 0$ and $4x^2 = 7$. If instead $a = 0$, the equation collapses to a linear equation $bx + c = 0$, since the x^2 term disappears entirely -- this is why the condition a not equal to 0 is essential to the definition.

1.2 Solving by Factorization and Completing the Square

Factorization method: write the equation in standard form $ax^2 + bx + c = 0$. Find two numbers r and s such that $r + s = b$ and $r*s = a*c$; then $ax^2 + bx + c$ splits into two linear factors, and setting each factor to zero gives the two roots. For example, to solve $3x^2 - 7x - 20 = 0$, since $-12 + 5 = -7$ and $-12 * 5 = -60 = 3*(-20)$, the equation becomes $3x^2 - 12x + 5x - 20 = 0$, which factors as $(x-4)(3x+5) = 0$, giving $x = 4$ or $x = -5/3$.

Completing the square method: shift the constant term to the right-hand side, then add the square of (half the coefficient of x) to both sides so the left side becomes a perfect square trinomial. For $x^2 - 3x - 4 = 0$, shifting gives $x^2 - 3x = 4$; adding $(-3/2)^2 = 9/4$ to both sides gives $(x - 3/2)^2 = 25/4$; taking the square root of both sides gives $x - 3/2 = +/-5/2$, so $x = 4$ or $x = -1$. If the coefficient of x^2 is not 1, every term is first divided by that coefficient before completing the square.

1.3 The Quadratic Formula

Applying the completing-the-square method to the general equation $ax^2 + bx + c = 0$ (dividing throughout by a, shifting c/a to the right, then adding $(b/2a)^2$

to both sides) leads to the general result $x = (-b \pm \sqrt{b^2 - 4ac}) / 2a$, a not equal to 0 -- known as the quadratic formula. It solves ANY quadratic equation directly from its coefficients a, b, c, without needing to find a factorization by inspection, which makes it especially useful when the roots are irrational or when factorization is not obvious. For example, solving $5x^2 - 9x - 2 = 0$ by substituting $a=5$, $b=-9$, $c=-2$ gives $x = (9 \pm \sqrt{81+40})/10 = (9 \pm 11)/10$, so $x = 2$ or $x = -1/5$.

1.4 Equations Reducible to Quadratic Form

Several equation types that do not initially look quadratic can be converted into one by a suitable substitution, solved as a quadratic in the new variable, and then converted back. Type (i): $ax^4 + bx^2 + c = 0$ -- substitute $y = x^2$ to get a quadratic in y, solve for y, then solve $x^2 = y$ for x. Type (ii): $a*p(x) + b/p(x) = c$ -- substitute $y = p(x)$ to clear the fraction into a quadratic in y. Type (iii) reciprocal equations: $a(x^2 + 1/x^2) + b(x + 1/x) + c = 0$ -- substitute $y = x + 1/x$, noting that $x^2 + 1/x^2 = y^2 - 2$, to reduce it to a quadratic in y.

Type (iv) exponential equations: the variable appears in the exponent, e.g. $5^{(1+x)} + 5^{(1-x)} = 26$; rewriting using exponent laws and substituting $y = 5^x$ converts it into a quadratic in y. Type (v): $(x+a)(x+b)(x+c)(x+d) = k$, where $a+b = c+d$ -- pairing the factors as $[(x+a)(x+b)][(x+c)(x+d)]$ and substituting y for the repeated quadratic expression that results converts the equation into a quadratic in y. In every type, the key skill is recognizing which substitution turns the given equation into a standard quadratic.

1.5 Radical Equations

An equation in which an expression involving the variable appears under a radical (square-root) sign is called a radical equation, e.g. $\sqrt{x+3} = x+1$. The general strategy is to isolate a radical on one side and square both sides to remove it, repeating if a second radical remains, until a polynomial (usually quadratic) equation results -- which is then solved by the usual methods.

Because squaring both sides of an equation can introduce extraneous roots (values that satisfy the squared equation but not the original one), every candidate solution to a radical equation MUST be substituted back into the original (unsquared) equation to check whether it genuinely satisfies it; any value that fails this check must be rejected. For example, solving sqrt(3x+7) = 2x+3 by squaring leads to $4x^2 + 9x + 2 = 0$, giving $x = -1/4$ or $x = -2$; checking both in the original equation determines which (if any) must be discarded as extraneous.

Important Definitions

What is a quadratic equation?

An equation containing the square of the unknown variable but no higher power, written in standard form as $ax^2 + bx + c = 0$ where a is not equal to 0.

What is a pure quadratic equation?

A quadratic equation in which the coefficient of x (that is, b) is zero, so it has the form $ax^2 + c = 0$, e.g. $x^2 - 16 = 0$.

What is the standard form of a quadratic equation?

The form $ax^2 + bx + c = 0$, with a, b, c real numbers and a not equal to 0, where a is the coefficient of x^2, b is the coefficient of x, and c is the constant term.

What is the quadratic formula?

$x = (-b \pm \sqrt{b^2 - 4ac}) / 2a$, a not equal to 0 -- a formula that gives both roots of any quadratic equation directly from its coefficients.

What is a reciprocal equation?

An equation that remains unchanged when x is replaced by 1/x, typically of the form $ax^4 + bx^3 + cx^2 + bx + a = 0$.

What is a radical equation?

An equation in which an expression containing the variable appears under a square-root (radical) sign, e.g. sqrt(x+3) = x+1.

What is an extraneous root?

A value obtained while solving an equation (typically after squaring both sides) that satisfies the transformed equation but does NOT satisfy the original equation, and must therefore be rejected.

What is the solution set of an equation?

The set containing all values of the variable that satisfy the equation -- for example {4, -5/3} for the equation $3x^2 - 7x - 20 = 0$.

What condition on a, b, c makes $ax^2+bx+c=0$ a genuine quadratic equation?

The condition a is not equal to 0 -- if $a = 0$, the x^2 term vanishes and the equation reduces to the linear equation $bx + c = 0$.

What substitution is used to solve $a(x^2 + 1/x^2) + b(x + 1/x) + c = 0$?

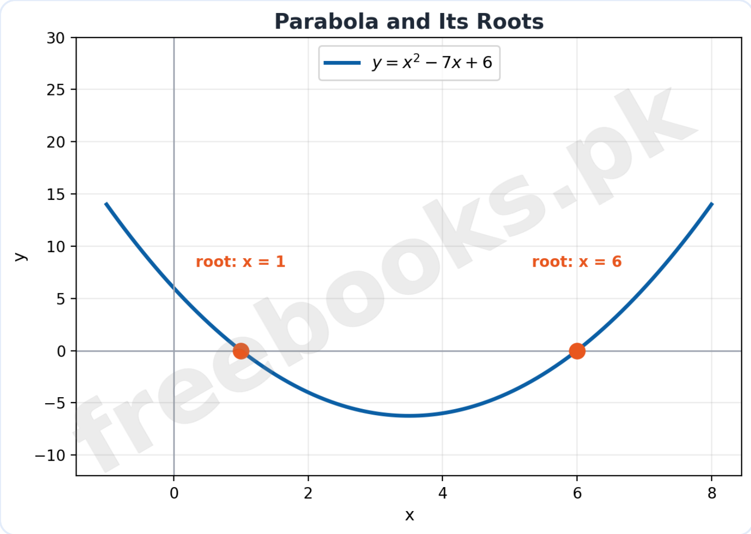
Let $y = x + 1/x$, so that $x^2 + 1/x^2 = y^2 - 2$; this converts the equation into a standard quadratic in y.

Key Facts and Relations

Topic	Key Fact / Relation
Standard form	$ax^2 + bx + c = 0$, a not equal to 0
Quadratic formula	$x = (-b \pm \sqrt{b^2 - 4ac}) / 2a$
Pure quadratic equation	$ax^2 + c = 0$ (i.e. $b = 0$)
Factorization condition	Find r, s such that $r + s = b$ and $r*s = a*c$
Completing the square step	Add $(b/2a)^2$ to both sides after shifting the constant term
Type $ax^4+bx^2+c=0$	Substitute $y = x^2$
Type $a*p(x)+b/p(x)=c$	Substitute $y = p(x)$
Reciprocal equation identity	$x^2 + 1/x^2 = (x + 1/x)^2 - 2$
Exponential equation substitution	Substitute $y = a^x$ for equations like $a^{(1+x)} + a^{(1-x)} = k$
Type $(x+a)(x+b)(x+c)(x+d)=k$	Valid when $a + b = c + d$; pair factors and substitute y for the repeated quadratic expression
Radical equation strategy	Isolate one radical, square both sides, repeat if needed, then solve the resulting polynomial equation
Extraneous root check	Every solution of a squared/radical equation must be verified in the ORIGINAL equation before being accepted

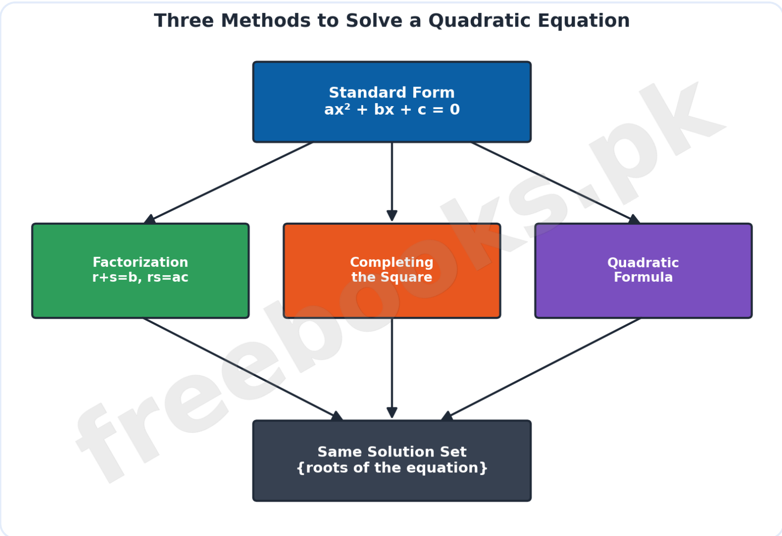
Diagrams

Parabola and Its Roots: A graph of y = x^2 - 7x + 6 showing the parabola crossing the x-axis at its two real roots, x = 1 and x = 6, illustrating that the roots of a quadratic equation are exactly the x-intercepts of its graph



Parabola and Its Roots

Three Methods to Solve a Quadratic Equation: A flow/comparison diagram showing the three standard solution methods -- Factorization, Completing the Square, and the Quadratic Formula -- branching from the standard form ax^2+bx+c=0, each leading to the same solution set



Three Methods to Solve a Quadratic Equation

Equations Reducible to Quadratic Form: A summary chart listing the five types of equations reducible to quadratic form (ax^4+bx^2+c=0, a*p(x)+b/p(x)=c, reciprocal equations, exponential equations, and (x+a)(x+b)(x+c)(x+d)=k) alongside the substitution used to reduce each to a standard quadratic

Equations Reducible to Quadratic Form

Type	Equation Form	Substitution
(i)	$ax^4 + bx^2 + c = 0$	$y = x^2$
(ii)	$a \cdot p(x) + b/p(x) = c$	$y = p(x)$
(iii) Reciprocal	$a(x^2+1/x^2) + b(x+1/x) + c = 0$	$y = x + 1/x$
(iv) Exponential	a^(1+x) + a^(1-x) = k	y = a^x
(v)	$(x+a)(x+b)(x+c)(x+d) = k$, $a+b=c+d$	y = repeated quadratic expr.

Equations Reducible to Quadratic Form

Short Questions & Answers

Write the standard form of a quadratic equation.

ax^2 + bx + c = 0, where a, b, c are real numbers and a is not equal to 0.

What makes an equation a pure quadratic equation?

The coefficient of x (b) is zero, so the equation has the form ax^2 + c = 0.

State the quadratic formula.

x = (-b +/- sqrt(b^2 - 4ac)) / 2a.

What substitution reduces ax^4 + bx^2 + c = 0 to a quadratic?

Let y = x^2, giving ay^2 + by + c = 0.

Why must roots of a radical equation be checked in the original equation?

Because squaring both sides can introduce extraneous roots that satisfy the squared equation but not the original one.

What identity connects x + 1/x and x^2 + 1/x^2?

x^2 + 1/x^2 = (x + 1/x)^2 - 2.

Long Questions & Answers

Explain, with the general method and a worked illustration, how a quadratic equation is solved by (a) factorization and (b) completing the square.

How does the factorization method work for solving a quadratic equation?

Write the equation in standard form $ax^2 + bx + c = 0$. Find two numbers r and s such that $r + s = b$ and r times s = a times c. Once found, rewrite the middle term bx as $rx + sx$, then factor the resulting four-term expression by grouping into two linear factors. Setting each linear factor equal to zero gives the two roots of the equation.

Work through an example of solving by factorization.

To solve $3x^2 - 7x - 20 = 0$: here $a=3$, $b=-7$, $c=-20$, so $a*c = -60$. The numbers -12 and 5 satisfy $-12+5=-7$ and $-12*5=-60$. Rewriting gives $3x^2 - 12x + 5x - 20 = 0$, which groups as $3x(x-4) + 5(x-4) = 0$, or $(x-4)(3x+5) = 0$. So $x = 4$ or $x = -5/3$, and the solution set is $\{-5/3, 4\}$.

How does the completing-the-square method work?

First make the coefficient of x^2 equal to 1 by dividing every term by a, if needed. Shift the constant term to the right-hand side. Add the square of half the coefficient of x to BOTH sides, which makes the left side a perfect square trinomial. Take the square root of both sides (remembering the plus-or-minus), then solve the resulting linear equation for x.

Work through an example of completing the square.

To solve $x^2 - 3x - 4 = 0$: shifting gives $x^2 - 3x = 4$. Adding $(-3/2)^2 = 9/4$ to both sides gives $(x - 3/2)^2 = 4 + 9/4 = 25/4$. Taking the square root gives $x - 3/2 = \pm 5/2$, so $x = 3/2 + 5/2 = 4$ or $x = 3/2 - 5/2 = -1$. The solution set is $\{-1, 4\}$.

Derive the quadratic formula from the standard form of a quadratic equation using the method of completing the square, and explain how it is used to solve equations that do not factor easily.

How is the quadratic formula derived?

Starting from $ax^2 + bx + c = 0$ (a not equal to 0), divide every term by a to get $x^2 + (b/a)x + c/a = 0$. Shift c/a to the right: $x^2 + (b/a)x = -c/a$. Add $(b/2a)^2$ to both sides to complete the square on the left, giving $(x + b/2a)^2 = (b^2 - 4ac)/4a^2$.

What is the final step of the derivation, and what formula results?

Taking the square root of both sides gives $x + b/2a = \pm \sqrt{b^2-4ac}/2a$. Isolating x gives the quadratic formula: $x = (-b \pm \sqrt{b^2 - 4ac}) / 2a$, valid for any a, b, c with a not equal to 0.

Why is the quadratic formula useful even when factorization is possible?

It solves any quadratic equation directly from its coefficients without needing to search for a factorization by inspection, which makes it essential when the roots are irrational, complex, or otherwise not obvious from the coefficients -- unlike factorization, it always works.

Work through an example using the quadratic formula.

To solve $5x^2 - 9x - 2 = 0$: here $a=5$, $b=-9$, $c=-2$. Substituting gives $x = (9 \pm \sqrt{81 - 4(5)(-2)})/10 = (9 \pm \sqrt{121})/10 = (9 \pm 11)/10$. So $x = 20/10 = 2$ or $x = -2/10 = -1/5$, giving the solution set $\{-1/5, 2\}$.

Multiple Choice Questions (MCQs)

The standard form of a quadratic equation is:

A. $ax + b = 0$

B. $ax^2 + bx + c = 0$, $a \neq 0$

C. $ax^3 + bx + c = 0$

D. $ax^2 + b = 0$ only

Answer: B. $ax^2 + bx + c = 0$, $a \neq 0$

Explanation: The standard form of a quadratic (second-degree) equation is $ax^2 + bx + c = 0$ with a not equal to 0.

If $b = 0$ in $ax^2 + bx + c = 0$, the equation is called:

A. A linear equation

B. A pure quadratic equation

C. A reciprocal equation

D. A radical equation

Answer: B. A pure quadratic equation

Explanation: When $b = 0$, the equation reduces to $ax^2 + c = 0$, called a pure quadratic equation.

The quadratic formula is:

A. $x = -b \pm \sqrt{(b^2-4ac)} / 2a$

B. $x = -b \pm \sqrt{(b^2+4ac)} / 2a$

C. $x = b \pm \sqrt{(b^2-4ac)} / a$

D. $x = -b \pm \sqrt{(b^2-4ac)} / a$

Answer: A. $x = -b \pm \sqrt{(b^2-4ac)} / 2a$

Explanation: The quadratic formula, derived by completing the square on $ax^2+bx+c=0$, is $x = (-b \pm \sqrt{(b^2-4ac)})/2a$.

To reduce $ax^4 + bx^2 + c = 0$ to a quadratic equation, the correct substitution is:

A. $y = x$

B. $y = x^2$

C. $y = 1/x$

D. $y = x^4$

Answer: B. $y = x^2$

Explanation: Substituting $y = x^2$ turns $ax^4+bx^2+c=0$ into the quadratic $ay^2+by+c=0$.

A value that satisfies a squared equation but not the original equation is called:

A. A pure root

B. A reciprocal root

C. An extraneous root

D. A rational root

Answer: C. An extraneous root

Explanation: An extraneous root satisfies the transformed (e.g. squared) equation but fails to satisfy the original equation, so it must be rejected.

For x + 1/x = y, the expression x^2 + 1/x^2 equals:

A. y^2

B. y^2 - 1

C. y^2 - 2

D. y^2 + 2

Answer: C. y^2 - 2

Explanation: Squaring x + 1/x = y gives x^2 + 2 + 1/x^2 = y^2, so x^2 + 1/x^2 = y^2 - 2.

Which method always works for solving any quadratic equation, factorable or not?

A. Factorization only

B. Completing the square only

C. The quadratic formula

D. Guessing

Answer: C. The quadratic formula

Explanation: Unlike factorization (which requires the equation to factor neatly), the quadratic formula solves any quadratic equation directly from its coefficients.

An equation containing the variable under a square-root sign is called:

A. A reciprocal equation

B. A radical equation

C. An exponential equation

D. A pure quadratic equation

Answer: B. A radical equation

Explanation: An equation with an expression under a radical (square-root) sign is called a radical equation.

In factorization by the ac-method for ax^2+bx+c=0, the two numbers r and s must satisfy:

A. r+s=a, rs=b

B. r+s=c, rs=a

C. r+s=b, rs=ac

D. r+s=ac, rs=b

Answer: C. r+s=b, rs=ac

Explanation: The two numbers r and s must satisfy $r+s=b$ and $r\times s=a\times c$ so that bx can be split into $rx+sx$ for grouping.

The roots of a quadratic equation $y = ax^2+bx+c$ correspond graphically to:

A. The y-intercept
B. The vertex only
C. The x-intercepts of the parabola
D. The slope of the curve

Answer: C. The x-intercepts of the parabola

Explanation: The real roots of $ax^2+bx+c=0$ are exactly the x-values where the parabola $y=ax^2+bx+c$ crosses the x-axis.

Quick Revision Summary

- A quadratic equation has the standard form $ax^2 + bx + c = 0$, $a \neq 0$
- $b = 0$ gives a pure quadratic equation $ax^2 + c = 0$
- Factorization: find r, s with $r+s=b$, $rs=ac$, then split and group
- Completing the square: shift constant, add (half coefficient of x)^2 to both sides
- Quadratic formula: $x = (-b \pm \sqrt{(b^2-4ac)}) / 2a$ — works for every quadratic equation
- $ax^4+bx^2+c=0$ reduces to a quadratic via $y=x^2$
- Reciprocal equations reduce via $y = x + 1/x$, using $x^2+1/x^2 = y^2-2$
- Exponential equations (variable in exponent) reduce via $y = a^x$
- $(x+a)(x+b)(x+c)(x+d)=k$ reduces via pairing when $a+b=c+d$
- Radical equations: isolate the radical, square, solve, then CHECK for extraneous roots
- Every squared/simplified solution must be verified in the original equation

Exam Tips

- Always rearrange an equation into standard form $ax^2+bx+c=0$ before choosing a solution method
- Try factorization first (fast); switch to the quadratic formula if a*c doesn't factor easily

- In completing the square, always add the same term to BOTH sides of the equation
- Memorize the quadratic formula exactly — a sign error in -b or ±√ is the most common mistake
- For any radical equation, substitute every candidate root back into the ORIGINAL equation before writing the final answer
- For reciprocal-type equations, remember $x^2+1/x^2 = (x+1/x)^2 - 2$, not $(x+1/x)^2$

Unit 2: Theory of Quadratic Equations

Building on Unit 1's solution methods, this unit studies quadratic equations more theoretically: it introduces the discriminant, which reveals the nature of a quadratic equation's roots WITHOUT actually solving the equation, and the cube roots of unity, a special set of three numbers (1, omega, omega-squared) with elegant algebraic properties used throughout higher algebra.

The unit then develops the deep relationship between a quadratic equation's roots and its coefficients -- allowing the sum and product of roots (and many other symmetric combinations of them) to be found without solving the equation at all -- and uses this relationship in reverse to construct new quadratic equations, including ones whose roots are transformations of another equation's roots. It closes with synthetic division (a fast method for dividing polynomials by linear factors, useful for solving cubic and quartic equations when some roots are already known), simultaneous equations in two variables, and real-life word problems that translate into quadratic equations.

Learning Objectives

- Define the discriminant (b^2 - 4ac) of a quadratic expression and find it for a given equation
- Use the discriminant to determine the nature of the roots of a quadratic equation without solving it
- Find the cube roots of unity and recognize the complex roots as omega and omega-squared
- Prove and apply the key properties of the cube roots of unity
- Find the sum and product of the roots of a quadratic equation from its coefficients, without solving it
- Find unknown values in a quadratic equation given a condition on the nature or relation of its roots
- Evaluate symmetric functions of the roots of a quadratic equation in terms of its coefficients

- Form a quadratic equation from given roots, including roots that are transformations of another equation's roots
- Use synthetic division to divide polynomials, evaluate unknowns, and solve cubic/quartic equations
- Solve a system of two equations in two variables, and apply quadratic equations to real-life word problems

Key Concepts

2.1 Discriminant and Nature of Roots

The two roots of ax^2 + bx + c = 0 (a not equal to 0) are (-b + sqrt(b^2-4ac))/2a and (-b - sqrt(b^2-4ac))/2a. The nature of these roots depends entirely on the value of the expression b^2 - 4ac, called the discriminant of the quadratic equation. To find it, simply identify a, b, c and substitute into b^2 - 4ac -- for example, for 2x^2 - 7x + 1 = 0, Disc. = (-7)^2 - 4(2)(1) = 49 - 8 = 41.

When a, b, c are rational numbers, the discriminant classifies the roots into four cases: if b^2-4ac > 0 and is a perfect square, the roots are rational (real) and unequal; if b^2-4ac > 0 but not a perfect square, the roots are irrational (real) and unequal; if b^2-4ac = 0, the roots are rational (real) and equal; if b^2-4ac < 0, the roots are imaginary (complex conjugates). This lets a problem determine the nature of an equation's roots by a single calculation, without applying the quadratic formula in full.

2.2 Cube Roots of Unity

The cube roots of unity are the three solutions of x^3 = 1, found by writing x^3 - 1 = 0 and factoring using a^3-b^3 = (a-b)(a^2+ab+b^2): (x-1)(x^2+x+1) = 0. Solving gives x = 1, or x = (-1 +/- i*sqrt(3))/2 using the quadratic formula on x^2+x+1=0. So the three cube roots of unity are 1, (-1+i*sqrt3)/2, and (-1-i*sqrt3)/2. If either complex root is called omega, the other is omega-squared.

Four key properties (all provable directly from the values above) make omega extremely useful in algebra: each complex cube root is the square of the other; the product of all three cube roots is 1, i.e. omega^3 = 1; each complex cube root is the reciprocal of the other, since omega * omega^2 = 1; and the sum of all three cube roots is zero, i.e. 1 + omega + omega^2 = 0 -- which gives the useful deductions omega + omega^2 = -1, 1+omega = -omega^2, and 1+omega^2 = -omega. Because omega^3 = 1, any higher power of omega can be reduced by dividing the exponent by 3 and keeping only the remainder, e.g. omega^7 = (omega^3)^2 * omega = omega.

2.3 Roots and Coefficients of a Quadratic Equation

For ax^2+bx+c=0 with roots alpha and beta, adding the two root expressions from the quadratic formula gives alpha+beta = -b/a, and multiplying them (using difference of squares) gives alpha*beta = c/a. Writing S for the sum and P for the product: S = -b/a = -(coefficient of x)/(coefficient of x^2), and P = c/a = (constant term)/(coefficient of x^2). This means the sum and product of the roots can be read directly off the coefficients, without solving the equation at all -- e.g. for 3x^2-5x+7=0, S = -(-5)/3 = 5/3 and P = 7/3.

This relationship is used in reverse to find unknown constants in an equation given a condition connecting its roots -- common conditions include: the sum of roots equals a multiple of the product; the sum of the squares of the roots equals a given number (using alpha^2+beta^2 = (alpha+beta)^2 - 2*alpha*beta); the roots differ by a given amount; the roots satisfy a linear relation such as 2*alpha+5*beta=7; or both the sum and product equal the same given number lambda. In every case, express S and P in terms of the unknown constant using the -b/a, c/a formulas, then use the given condition to form and solve an equation for that constant.

2.4 Symmetric Functions of the Roots

A symmetric function of alpha and beta is an expression whose value is unchanged when alpha and beta are swapped -- for example alpha^2+beta^2, since beta^2+alpha^2 gives the same value. Because S = alpha+beta and P = alpha*beta are themselves symmetric functions, EVERY symmetric function of the roots can be rewritten purely in terms of S and P (hence purely in terms of the coefficients a, b, c), without ever finding alpha and beta individually.

Common identities used for this rewriting: alpha^2+beta^2 = (alpha+beta)^2 - 2*alpha*beta = S^2-2P; alpha^2*beta+alpha*beta^2 = alpha*beta(alpha+beta) = PS; 1/alpha + 1/beta = (alpha+beta)/(alpha*beta) = S/P. For example, if alpha, beta are the roots of 2x^2+3x+4=0, then S=-3/2 and P=2, so alpha^2+beta^2 = (-3/2)^2 - 2(2) = 9/4-4 = -7/4.

2.5 Forming a Quadratic Equation from Its Roots

If alpha and beta are the required roots, then x=alpha and x=beta, so (x-alpha)(x-beta)=0, which expands to x^2 - (alpha+beta)x + alpha*beta = 0, or simply x^2 - Sx + P = 0. This single formula lets any quadratic equation be reconstructed once its sum and product of roots are known -- e.g. roots 3 and 4 give S=7, P=12, so the equation is x^2-7x+12=0.

The same formula solves the harder problem of forming an equation whose roots are some TRANSFORMATION of another equation's roots (e.g. 2*alpha+1 and 2*beta+1, or alpha^2 and beta^2, or 1/alpha and 1/beta) -- first find the original equation's S and P from its coefficients, then compute the NEW sum and product using the transformation, and finally substitute the new S and P into x^2-Sx+P=0.

2.6 Synthetic Division

Synthetic division is a fast shortcut for dividing a polynomial P(x) by a linear divisor x-a: write the polynomial's coefficients in a row (using 0 for any missing power of x), place a on the left, bring down the first coefficient, then repeatedly multiply the most recent result by a and add it to the next coefficient. The final row gives the coefficients of the quotient (one degree lower than the dividend) and the last entry is the remainder.

Synthetic division has five main uses: finding the quotient and remainder of a polynomial division directly; finding unknown constants when a zero of the polynomial is given (the remainder must be 0); finding unknowns when specific linear factors are given (each factor's root gives remainder 0, producing simultaneous equations); solving a cubic equation when one root is already known (dividing it out leaves a quadratic 'depressed equation' solvable by the usual methods); and solving a quartic (biquadratic) equation when two of its real roots are known (dividing out both roots in succession leaves a quadratic).

2.7 Simultaneous Equations and Word Problems

A system of simultaneous equations is a set of equations sharing a common solution; the solution set consists of every ordered pair (x,y) satisfying all the equations at once. When ONE equation is linear and the other quadratic, solve the linear equation for y in terms of x, substitute into the quadratic equation to get a new quadratic in x alone, solve it, then find the matching y-values. When BOTH equations are quadratic, a common strategy is to eliminate one squared term by suitable multiplication and subtraction (similar to elimination for linear systems), which often produces a factorable relation between x and y that can be substituted back into either original equation.

Many real-life problems -- involving consecutive numbers, rectangle dimensions, or coordinates of a point -- translate directly into a quadratic equation (or a system of two equations) once the unknown quantities are represented by variables and the given conditions are written as equations. The roots of the resulting equation(s) then answer the original question, though a root must sometimes be rejected if it doesn't make physical sense (e.g. a negative length).

Important Definitions

What is the discriminant of a quadratic equation?

The expression b^2 - 4ac for the equation ax^2+bx+c=0, whose sign and whether it is a perfect square determine the nature of the roots.

What are the cube roots of unity?

The three solutions of x^3 = 1, namely 1, omega = (-1+i*sqrt3)/2, and omega^2 = (-1-i*sqrt3)/2.

What is omega (ω) in the context of cube roots of unity?

One of the two complex cube roots of unity; the other complex root is then omega-squared, and together with 1 they are the three cube roots of unity.

What is a symmetric function of the roots of an equation?

An expression involving the roots alpha and beta whose value stays the same when alpha and beta are interchanged, e.g. alpha^2+beta^2 or alpha*beta.

What is synthetic division?

A shortcut method for dividing a polynomial by a linear divisor x-a, using only the polynomial's coefficients, that gives the quotient's coefficients and the remainder directly.

What is a depressed equation?

The lower-degree equation left over after synthetic division removes one known root from a cubic or quartic equation, which can then be solved by standard methods.

What is a system of simultaneous equations?

A set of equations that share a common solution set, consisting of every ordered pair (or set of values) that satisfies all the equations at once.

If S is the sum and P is the product of the roots of ax^2+bx+c=0, what are S and P in terms of a, b, c?

S = -b/a and P = c/a.

What formula reconstructs a quadratic equation from its sum S and product P of roots?

x^2 - Sx + P = 0.

What is the formula a^3 - b^3 used to derive the cube roots of unity?

a^3 - b^3 = (a-b)(a^2+ab+b^2), applied to x^3-1=0 as (x-1)(x^2+x+1)=0.

Key Facts and Relations

Topic	Key Fact / Relation
Discriminant	Disc. = b^2 - 4ac
Roots rational & unequal	b^2-4ac > 0 and a perfect square
Roots irrational & unequal	b^2-4ac > 0 and not a perfect square
Roots rational & equal	b^2-4ac = 0
Roots imaginary (complex conjugates)	b^2-4ac < 0
Cube roots of unity	1, (-1+i√3)/2 = ω, (-1-i√3)/2 = ω^2
Key omega identities	$1+\omega+\omega^2=0$, $\omega^3=1$, $\omega\cdot\omega^2=1$

Topic	Key Fact / Relation
Sum of roots	$S = \alpha+\beta = -b/a$
Product of roots	$P = \alpha\beta = c/a$
Equation from S and P	$x^2 - Sx + P = 0$
Common symmetric identity	$\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta = S^2-2P$
Synthetic division	Divide P(x) by (x-a): bring down leading coefficient, repeatedly ×a and add

Diagrams

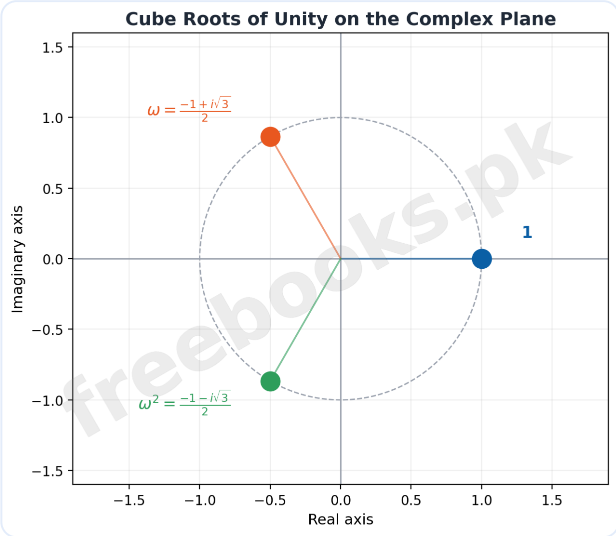
Discriminant and Nature of Roots: A summary chart showing the four cases of the discriminant b^2-4ac (positive perfect square, positive non-perfect-square, zero, negative) alongside the corresponding nature of the roots in each case

Discriminant and Nature of Roots

Discriminant (b^2-4ac)	Nature of Roots
Positive & perfect square	Rational (real) and unequal
Positive & not a perfect square	Irrational (real) and unequal
Equal to zero	Rational (real) and equal
Negative	Imaginary (complex conjugates)

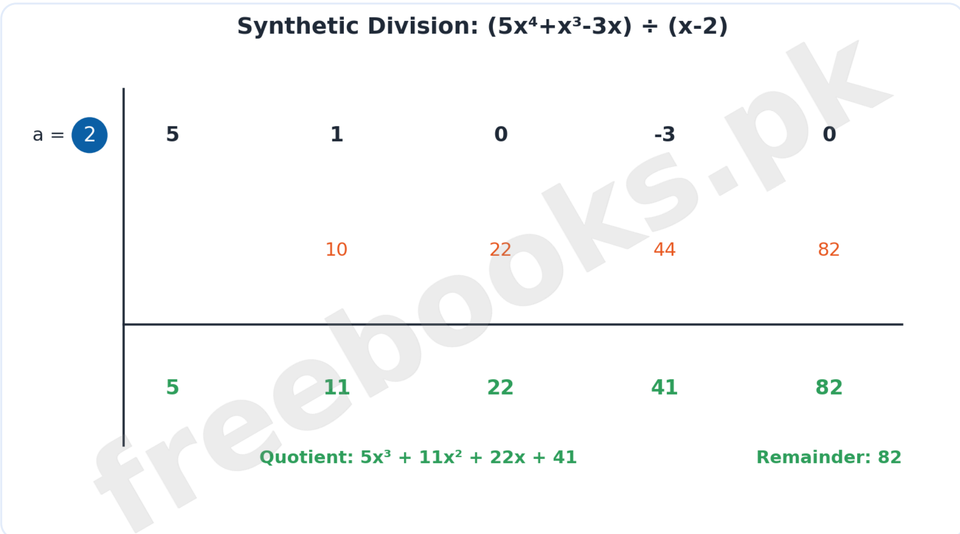
Discriminant and Nature of Roots

Cube Roots of Unity on the Complex Plane: An Argand diagram (complex plane) showing the three cube roots of unity -- 1, omega, and omega-squared -- as three points equally spaced (120 degrees apart) on the unit circle



Cube Roots of Unity on the Complex Plane

Synthetic Division Layout: A worked diagram showing the synthetic division grid for dividing 5x^4+x^3-3x by x-2, illustrating the bring-down, multiply, and add steps that produce the quotient's coefficients and the remainder



Synthetic Division Layout

Short Questions & Answers

Write the formula for the discriminant of ax^2+bx+c=0.

Discriminant = b^2 - 4ac.

If the discriminant of a quadratic equation is zero, what is the nature of its roots?

The roots are rational (real) and equal.

Name the three cube roots of unity.

1, $(-1+i\sqrt{3})/2$ ($=\omega$), and $(-1-i\sqrt{3})/2$ ($=\omega^2$).

State the value of $1 + \omega + \omega^2$.

0 (the sum of all three cube roots of unity is zero).

Write S and P in terms of a, b, c for $ax^2+bx+c=0$.

S (sum of roots) = $-b/a$, and P (product of roots) = c/a.

What formula is used to form a quadratic equation from its sum and product of roots?

$x^2 - Sx + P = 0$.

Long Questions & Answers

Explain how the discriminant of a quadratic equation determines the nature of its roots, and describe the cube roots of unity together with their key properties.

What is the discriminant, and how is it calculated?

For $ax^2+bx+c=0$ (a not equal to 0), the discriminant is the expression b^2-4ac. It is calculated by identifying a, b, c from the equation's standard form and substituting them in -- e.g. for $2x^2-7x+1=0$, Disc. = $(-7)^2-4(2)(1) = 49-8 = 41$.

How does the discriminant reveal the nature of the roots without solving the equation?

For rational a, b, c: if the discriminant is positive and a perfect square, the roots are rational (real) and unequal; if positive but not a perfect square, the roots are irrational (real) and unequal; if the discriminant is exactly zero, the roots are rational (real) and equal; and if the discriminant is negative, the roots are imaginary (complex conjugates).

How are the cube roots of unity derived?

Setting $x^3=1$ gives $x^3-1=0$, which factors (using $a^3-b^3=(a-b)(a^2+ab+b^2)$) as $(x-1)(x^2+x+1)=0$. Solving $x-1=0$ gives $x=1$; solving $x^2+x+1=0$ by the quadratic formula gives $x=(-1\pm i\sqrt{3})/2$. So the three cube roots of unity are 1, (-1+i√3)/2, and (-1-i√3)/2; naming one complex root omega makes the other omega-squared.

What are the key properties of the cube roots of unity?

Each complex cube root is the square of the other; their product is 1 (so omega^3=1); each complex cube root is the reciprocal of the other (omega·omega^2=1); and their sum is zero (1+omega+omega^2=0), giving the useful deductions omega+omega^2=-1, 1+omega=-omega^2, and 1+omega^2=-omega.

How are higher powers of omega simplified?

Since omega^3=1, any power of omega can be reduced by dividing its exponent by 3 and keeping only the remainder as the effective power -- e.g. omega^7 = (omega^3)^2·omega = (1)^2·omega = omega, and omega^{63} = (omega^3)^{21} = 1.

Explain the relationship between the roots and coefficients of a quadratic equation, and describe how this relationship is used to form new quadratic equations and solve polynomial equations by synthetic division.

What is the relationship between a quadratic equation's roots and its coefficients?

For ax^2+bx+c=0 with roots alpha and beta, the sum of the roots S = alpha+beta = -b/a, and the product of the roots P = alpha·beta = c/a. This means S and P can be read directly from the coefficients without ever solving the equation.

How are unknown constants found when a condition on the roots is given?

Express S and P in terms of the unknown constant using S=-b/a and P=c/a, then substitute into the given condition (e.g. sum equals a multiple of the product, or the roots differ by a given amount) to form and solve an equation for that constant.

How is a new quadratic equation formed from a known sum and product of roots?

Using the formula $x^2 - Sx + P = 0$, where S and P are the sum and product of the desired roots -- e.g. roots 3 and 4 give $S=7$, $P=12$, so the equation is $x^2-7x+12=0$. For roots that are a transformation of another equation's roots (like $2\alpha+1$, $2\beta+1$), first find the original S and P, then compute the transformed sum and product before substituting.

What is synthetic division, and what does its result represent?

A shortcut method for dividing a polynomial $P(x)$ by a linear divisor $x-a$, using only its coefficients: bring down the leading coefficient, then repeatedly multiply the latest result by a and add it to the next coefficient. The final numbers give the quotient's coefficients (one degree lower than the dividend), and the very last entry is the remainder.

How is synthetic division used to solve a cubic or quartic equation when some roots are known?

Each known root is divided out in turn using synthetic division (its remainder should be zero, confirming it is genuinely a root), which reduces the equation's degree by one each time. Once enough roots have been divided out, the remaining 'depressed equation' is a quadratic, solvable by factorization or the quadratic formula to find the rest of the roots.

Multiple Choice Questions (MCQs)

The discriminant of $ax^2+bx+c=0$ is:

A. b^2+4ac
B. b^2-4ac
C. $4ac-b^2$
D. $b-4ac$
Answer: B. b^2-4ac
Explanation: The discriminant is defined as b^2-4ac for the standard quadratic equation $ax^2+bx+c=0$.

If the discriminant of a quadratic equation is negative, its roots are:

A. Rational and equal

B. Irrational and unequal

C. Imaginary (complex conjugates)

D. Rational and unequal

Answer: C. Imaginary (complex conjugates)

Explanation: A negative discriminant means the square root involves a negative number, giving imaginary (complex conjugate) roots.

The three cube roots of unity are the solutions of:

A. $x^2 = 1$

B. $x^3 = 1$

C. $x^3 = -1$

D. $x^2 = -1$

Answer: B. $x^3 = 1$

Explanation: The cube roots of unity are, by definition, the three solutions of $x^3 = 1$.

If omega is a complex cube root of unity, then 1 + omega + omega^2 equals:

A. 1

B. -1

C. 0

D. 3

Answer: C. 0

Explanation: The sum of all three cube roots of unity (1, omega, omega-squared) is always zero.

For ax^2+bx+c=0 with roots alpha, beta, the product alpha·beta equals:

A. -b/a

B. b/a

C. c/a

D. -c/a

Answer: C. c/a

Explanation: The product of the roots of a quadratic equation equals c/a (constant term over coefficient of x^2).

The quadratic equation with sum of roots S and product of roots P is:

A. $x^2+Sx+P=0$

B. $x^2-Sx+P=0$

C. $x^2-Sx-P=0$

D. $x^2+Sx-P=0$

Answer: B. $x^2-Sx+P=0$

Explanation: A quadratic equation can be reconstructed from its roots' sum and product using $x^2 - Sx + P = 0$.

alpha^2 + beta^2 can be rewritten in terms of S=alpha+beta and P=alpha·beta as:

A. $S^2 + 2P$

B. $S^2 - 2P$

C. $S^2 - P$

D. $2S - P$

Answer: B. $S^2 - 2P$

Explanation: Since (alpha+beta)^2 = alpha^2+2*alpha*beta+beta^2, rearranging gives alpha^2+beta^2 = (alpha+beta)^2 - 2*alpha*beta = S^2-2P.

In synthetic division of P(x) by (x-a), the last number obtained represents the:

A. Quotient

B. Leading coefficient

C. Remainder

D. Value of a

Answer: C. Remainder

Explanation: The last entry in a synthetic division row is always the remainder of the division.

A depressed equation is obtained after synthetic division by:

A. Adding a known root back

B. Dividing out one known root of a cubic/quartic equation

C. Multiplying two equations

D. Taking the discriminant

Answer: B. Dividing out one known root of a cubic/quartic equation

Explanation: Dividing out a known root via synthetic division reduces a cubic or quartic equation's degree, leaving a lower-degree 'depressed equation'.

In a system where one equation is linear and the other quadratic, the standard solving strategy is:
A. Add the two equations directly
B. Solve the linear equation for one variable and substitute into the quadratic
C. Take the discriminant of both
D. Graph only
Answer: B. Solve the linear equation for one variable and substitute into the quadratic
Explanation: The linear equation is solved for one variable in terms of the other, then substituted into the quadratic equation to get a solvable equation in one variable.

Quick Revision Summary

- Discriminant = b^2 - 4ac; its sign and perfect-square status determine the nature of the roots
- b^2-4ac>0 & perfect square: rational, unequal | not perfect square: irrational, unequal
- b^2-4ac=0: rational, equal | b^2-4ac<0: imaginary (complex conjugates)
- Cube roots of unity: 1, $\omega=(-1+i\sqrt{3})/2$, $\omega^2=(-1-i\sqrt{3})/2$, from $x^3-1=0$
- Key omega facts: $1+\omega+\omega^2=0$, $\omega^3=1$, $\omega\cdot\omega^2=1$, each root is the square of the other
- Sum of roots S=-b/a, Product of roots P=c/a — no need to solve the equation
- New equation from roots: $x^2 - Sx + P = 0$
- Symmetric functions ($\alpha^2+\beta^2$, $1/\alpha+1/\beta$, etc.) can always be rewritten using S and P
- Synthetic division: divide a polynomial by (x-a) using only coefficients; last number = remainder
- A known root removed via synthetic division leaves a lower-degree depressed equation
- Simultaneous equations: substitute the linear equation into the quadratic one, or eliminate a squared term

Exam Tips

- Compute the discriminant first before attempting to factor — it tells you in advance whether factorization will even work nicely
- Memorize the four discriminant cases exactly; exam questions often ask to state the nature of roots WITHOUT solving
- For omega problems, always try to reduce using $\omega^3=1$ and $1+\omega+\omega^2=0$ before expanding anything directly
- When forming an equation from transformed roots, compute the original S and P first, then transform them — don't try to find alpha and beta explicitly
- In synthetic division, always include a 0 placeholder for any missing power of x in the dividend
- For real-life word problems, always check whether a negative or non-physical root should be rejected before giving the final answer

Unit 3: Variations

This unit builds a toolkit for comparing quantities and describing how one quantity changes in response to another. It starts with ratio and proportion -- the language used to compare two or more quantities of the same kind -- and extends this into direct and inverse variation, which describe how one quantity grows or shrinks as another one changes, a relationship that appears constantly in physics, engineering, and everyday life (speed and time, pressure and depth, force and distance).

The unit then develops five classical theorems on proportions (invertendo, alternendo, componendo, dividendo, and componendo-dividendo) that let complicated-looking ratio equations be manipulated and simplified quickly, along with the k-method, a systematic technique for proving that two such ratio expressions are equal. It closes with joint variation -- where a quantity depends on several others at once, some directly and some inversely -- and real-life problems (Hooke's law, Newton's law of gravitation, kinetic energy) that are solved by translating a verbal description of a relationship into a variation equation.

Learning Objectives

- Define ratio, proportion, and variation (direct and inverse), and solve basic problems involving them
- Find the third, fourth, mean, and continued proportional of given quantities
- State and apply the theorems of invertendo, alternendo, componendo, dividendo, and componendo-dividendo
- Use the k-method to prove conditional equalities involving proportions
- Define joint variation and solve problems where a quantity varies jointly with several others
- Translate real-life word problems (physics, geometry, economics) into variation equations and solve them

Key Concepts

3.1 Ratio, Proportion, and Variation

A ratio is a relation between two quantities of the same kind (measured in the same unit), written a:b or as the fraction a/b, where b is not zero. The order of the terms matters -- in a:b, a is called the antecedent and b the consequent -- and a ratio itself has no units, since the units cancel. A proportion is a statement that two ratios are equal: if a:b = c:d, this is written a:b::c:d, where a and d are called the extremes and b, c are called the means. The key working rule is that the product of the extremes equals the product of the means, i.e. ad = bc -- this single fact solves most proportion problems, including finding an unknown term or checking whether four given quantities are in proportion.

Variation describes how one quantity changes as another changes, and comes in two basic forms. In direct variation, an increase (or decrease) in one quantity causes a corresponding increase (or decrease) in the other: if y varies directly as x, this is written $y \propto x$, meaning y = kx for some constant k (called the constant of variation) that is not zero. In inverse variation, an increase in one quantity causes a decrease in the other: if y varies inversely as x, this is written $y \propto 1/x$, meaning y = k/x, or equivalently xy = k. In both cases, once k is found from one known pair of values, the same equation predicts y for any other value of x (or vice versa) -- variation can also apply to powers of a variable, such as $y \propto x^2$ or $y \propto x^3$, following exactly the same method.

3.2 Third, Fourth, Mean, and Continued Proportional

Given two quantities a and b, if a third quantity c is chosen so that a:b::b:c, then c is called the third proportional to a and b -- notice b is repeated as both a mean. Given three quantities a, b, c, if a fourth quantity d is chosen so that a:b::c:d, then d is called the fourth proportional. Both are found the same way: set up the proportion, apply 'product of extremes = product of means', and solve the resulting equation for the unknown term.

If three quantities a, b, c satisfy a:b::b:c (the same pattern as the third proportional), then b itself -- the repeated middle term -- is called the mean proportional between a and c, and satisfies $b^2 = ac$, so $b = \pm\sqrt{ac}$. When three quantities a, b, c are related this same way (a:b::b:c), with a first, b as the mean, and c third, they are together said to be in continued proportion; this idea can be extended to four or more quantities all connected by equal successive ratios.

3.3 Theorems on Proportions

If a:b = c:d is a proportion, several useful new proportions can be deduced from it by standard theorems, each proved by writing a/b = c/d and manipulating the fraction algebraically. Invertendo: if a:b = c:d, then b:a = d:c (simply flip both ratios). Alternendo: if a:b = c:d, then a:c = b:d (swap the middle terms). Componendo: if a:b = c:d, then (a+b):b = (c+d):d, and equivalently a:(a+b) = c:(c+d) (add the denominator to the numerator on each side).

Dividendo: if a:b = c:d, then (a-b):b = (c-d):d, and equivalently a:(a-b) = c:(c-d) (subtract the denominator from the numerator on each side). Componendo-Dividendo, the most powerful of the five and the one most frequently tested: if a:b = c:d, then (a+b):(a-b) = (c+d):(c-d). This is especially useful for solving equations that already have a 'sum over difference' shape, or for combining two componendo results and two dividendo results to eliminate variables and reach a much simpler ratio, as when proving results like 3m+7n:3m-7n = 3p+7q:3p-7q from m:n=p:q.

3.4 Joint Variation

Joint variation combines direct and inverse variation of two or more variables into a single relationship. If y varies directly as x and inversely as z, this is written together as $y \propto x/z$, meaning $y = kx/z$ for some constant k. More generally, a quantity can vary directly as the product of several quantities and inversely as the product of several others, all combined into one proportionality and one constant k -- for example, by Newton's law of gravitation, the gravitational force G between two bodies varies directly as the product of their masses m1, m2 and inversely as the square of the distance d between them, written $G \propto (m1 \cdot m2)/d^2$, i.e. $G = k(m1 \cdot m2)/d^2$.

Joint variation problems are solved in three steps: first, translate the verbal description into a proportionality statement (e.g. 'y varies jointly as x^2 and z' becomes $y \propto x^2 \cdot z$); second, insert the constant of variation to get an equation ($y = k \cdot x^2 \cdot z$) and substitute one complete set of known values to solve for k;

third, substitute the now-known k back into the equation, along with any new values given, to find the requested unknown.

3.5 K-Method

The k-method is a systematic technique for proving that two more complicated ratio expressions are equal, given that a:b = c:d (or a:b = c:d = e:f for three ratios). The method sets each of the given equal ratios to a single constant k -- e.g. a/b = c/d = k -- which gives a = bk and c = dk (and e = fk, if a third ratio is involved). These substitutions are then made into the Left Hand Side and Right Hand Side of whatever is to be proved; since both sides are rewritten purely in terms of b, d (and f) and the same k, common factors cancel and both sides simplify to the identical expression, completing the proof.

The k-method works because it replaces two 'unknown but equal ratios' with a single shared variable k, turning what looks like an abstract algebraic identity into straightforward substitution and simplification. It is the standard tool for proving conditional equalities of the form 'if a:b=c:d, then (some expression in a,b) = (the same expression in c,d)', and is far faster and more reliable than trying to manipulate the original ratio equation directly with cross-multiplication.

3.6 Real-Life Applications of Variation

Many real-life laws in physics, engineering, and economics are stated as variation relationships and solved using the same three-step method (translate to a proportionality, find k from known values, then predict a new value). Examples include Hooke's law (the force needed to stretch a spring varies directly as its elongation), the strength of a rectangular beam (varies directly as its breadth and the square of its depth), the intensity of light (varies inversely as the square of the distance from the source), and the kinetic energy of a moving body (varies jointly as its mass and the square of its velocity).

The general procedure: read the problem carefully to identify which quantities vary directly and which vary inversely (this determines whether each variable appears in the numerator or denominator of the k-expression); write the joint/direct/inverse variation equation with the constant k; substitute the first complete set of given values to solve for k; then substitute the new given values into the equation (with k now known) to find the requested quantity. Always check that the final answer makes physical sense for the situation described.

Important Definitions

What is a ratio?

A relation between two quantities of the same kind (measured in the same unit), written a:b or a/b, with no units of its own.

What is a proportion?

A statement that two ratios are equal, written a:b::c:d, where a, d are the extremes and b, c are the means, satisfying ad = bc.

What is direct variation?

A relationship where an increase (or decrease) in one quantity causes a corresponding increase (or decrease) in another, written $y \propto x$ or $y = kx$.

What is inverse variation?

A relationship where an increase in one quantity causes a decrease in another, written $y \propto 1/x$ or $y = k/x$, i.e. $xy = k$.

What is the third proportional to a and b?

The quantity c such that a:b::b:c.

What is the mean proportional between a and c?

The quantity b such that a:b::b:c, satisfying $b^2 = ac$, so $b = \pm\sqrt{(ac)}$.

What is joint variation?

A combination of direct and inverse variation of two or more variables into a single relationship, e.g. $y \propto x/z$, i.e. $y = kx/z$.

What is the constant of variation?

The non-zero constant k in a variation equation (such as $y = kx$ or $y = k/x$) that stays fixed for a given relationship.

State the componendo-dividendo theorem.

If a:b = c:d, then (a+b):(a-b) = (c+d):(c-d).

What is the k-method?

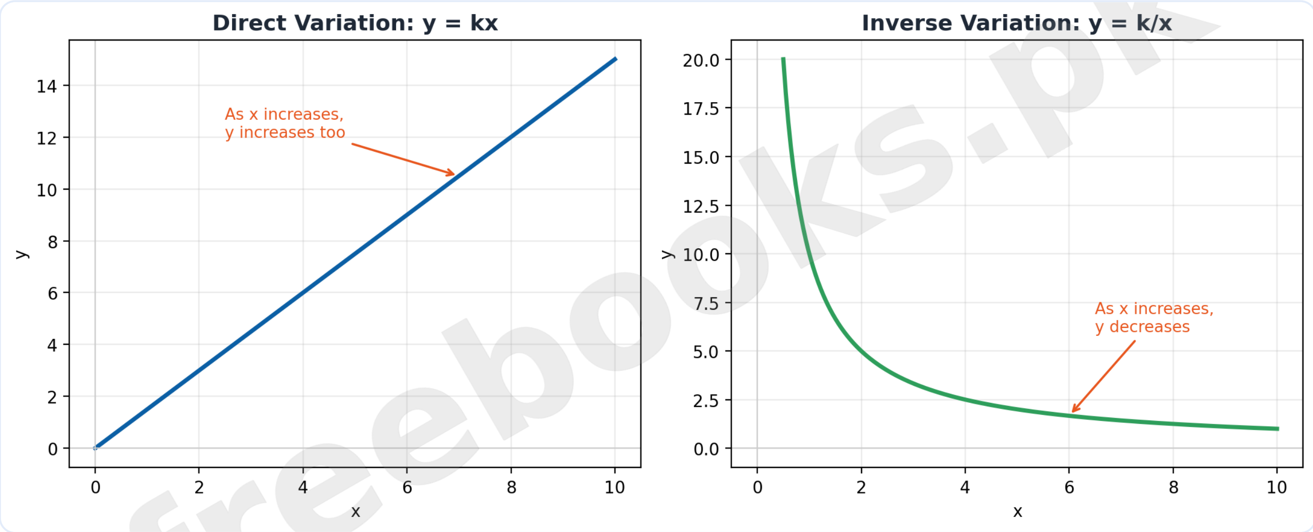
A technique for proving ratio identities by setting each side of a given proportion equal to a constant k, substituting, and simplifying both sides to the same expression.

Key Facts and Relations

Topic	Key Fact / Relation
Proportion rule	$a:b::c:d \Rightarrow ad = bc$ (product of extremes = product of means)
Direct variation	$y \propto x \Rightarrow y = kx$
Inverse variation	$y \propto 1/x \Rightarrow y = k/x$, i.e. $xy = k$
Third proportional	$a:b::b:c \Rightarrow c = b^2/a$
Mean proportional	$a:b::b:c \Rightarrow b = \pm\sqrt{(ac)}$
Invertendo	$a:b=c:d \Rightarrow b:a=d:c$
Alternendo	$a:b=c:d \Rightarrow a:c=b:d$
Componendo	$a:b=c:d \Rightarrow (a+b):b=(c+d):d$
Dividendo	$a:b=c:d \Rightarrow (a-b):b=(c-d):d$
Componendo-Dividendo	$a:b=c:d \Rightarrow (a+b):(a-b)=(c+d):(c-d)$
Joint variation	$y \propto x/z \Rightarrow y = kx/z$
K-method setup	$a/b = c/d = k \Rightarrow a=bk, c=dk$

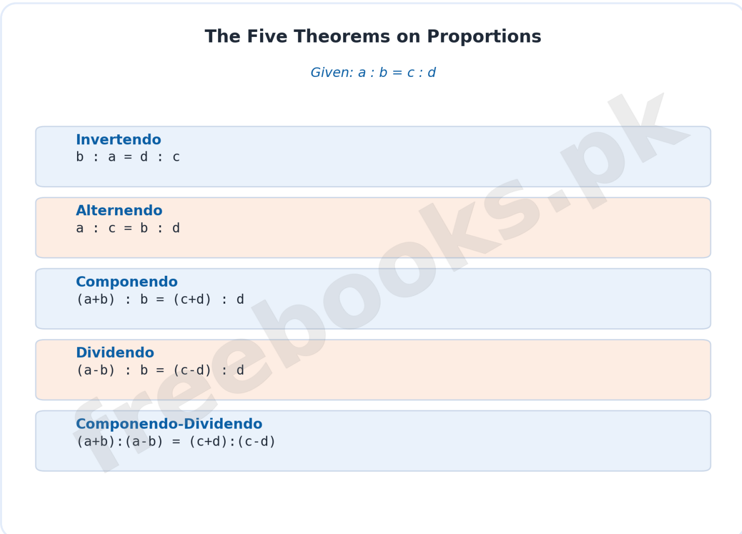
Diagrams

Direct vs Inverse Variation: Side-by-side plots of $y=kx$ (a straight line through the origin, direct variation) and $y=k/x$ (a hyperbola, inverse variation), showing how y changes as x increases in each case



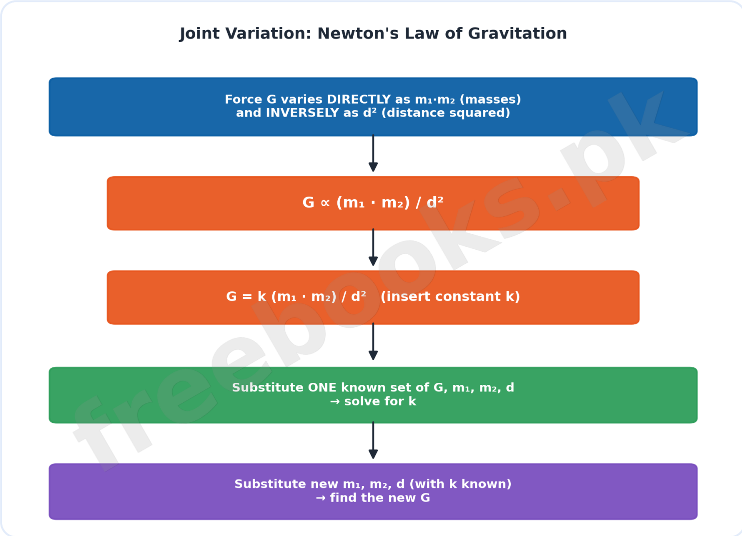
Direct vs Inverse Variation

The Five Theorems on Proportions: A summary chart listing invertendo, alternendo, componendo, dividendo, and componendo-dividendo alongside the transformation each theorem applies to a:b=c:d



The Five Theorems on Proportions

Joint Variation Flow: A flow diagram showing Newton's law of gravitation as a worked joint-variation example: force G varies directly as the product of two masses and inversely as the square of the distance between them



Joint Variation Flow

Short Questions & Answers

Define ratio.

A relation between two quantities of the same kind, measured in the same unit, written a:b or a/b, with no units.

In the proportion a:b::c:d, which terms are the extremes and which are the means?

a and d are the extremes; b and c are the means.

Write the mathematical form of 'y varies directly as x'.

$y \propto x$, i.e. $y = kx$, where k is the constant of variation.

Write the mathematical form of 'y varies inversely as x'.

$y \propto 1/x$, i.e. $y = k/x$, so $xy = k$.

State the componendo theorem.

If a:b = c:d, then (a+b):b = (c+d):d.

What does the k-method substitute for a given equal ratio a/b = c/d?

It sets $a/b = c/d = k$, giving $a = bk$ and $c = dk$, which are then substituted into both sides of the expression to be proved.

Long Questions & Answers

Explain ratio, proportion, and the two basic types of variation, and describe how the third, fourth, and mean proportional of given quantities are found.

What is a ratio, and what rule governs a proportion?

A ratio a:b (or a/b) compares two quantities of the same kind and has no units. A proportion states that two ratios are equal, a:b::c:d, where a, d are the extremes and b, c are the means; the defining rule is that the product of the extremes equals the product of the means, $ad = bc$, which is used to find an unknown term in any proportion.

What is direct variation, and how is it written?

Direct variation describes a relationship where an increase in one quantity produces a corresponding increase in another (and a decrease produces a decrease). If y varies directly as x, this is written $y \propto x$, meaning $y = kx$ for a non-zero constant k called the constant of variation, found by substituting one known pair of x, y values.

What is inverse variation, and how is it written?

Inverse variation describes a relationship where an increase in one quantity produces a decrease in the other. If y varies inversely as x, this is written $y \propto 1/x$, meaning $y = k/x$, or equivalently $xy = k$, with k again found from one known pair of values before predicting any other pair.

How is the third proportional to two quantities found?

For quantities a and b, the third proportional c satisfies a:b::b:c. Writing this as a fraction, $a/b = b/c$, cross-multiplying gives $ac = b^2$, so $c = b^2/a$ -- e.g. the third proportional to 6 and 12 is $12^2/6 = 24$.

How are the fourth proportional and the mean proportional found?

For a, b, c, the fourth proportional d satisfies a:b::c:d, so $ad = bc$, giving $d = bc/a$. The mean proportional between a and c is the quantity b satisfying a:b::b:c, so b^2

= ac, giving b = ±√(ac) -- both are solved by writing out the proportion and applying 'product of extremes = product of means'.

State the five theorems on proportions and the k-method, and explain how joint variation is used to solve real-life problems.

What do the theorems of invertendo and alternendo state?

If a:b = c:d, invertendo gives b:a = d:c (flipping both ratios), and alternendo gives a:c = b:d (swapping the middle terms b and c). Both are proved directly by manipulating the fraction a/b = c/d.

What do the theorems of componendo, dividendo, and componendo-dividendo state?

If a:b = c:d: componendo gives (a+b):b = (c+d):d; dividendo gives (a-b):b = (c-d):d; and combining both, componendo-dividendo gives (a+b):(a-b) = (c+d):(c-d) -- the most powerful of the five, often used to simplify equations already in a sum-over-difference form.

What is the k-method, and how is it used to prove a ratio identity?

Given a:b = c:d, the k-method sets a/b = c/d = k, so a = bk and c = dk. Substituting these into both the Left Hand Side and Right Hand Side of the expression to be proved lets the common factors b and d cancel, reducing both sides to the same expression in k alone and completing the proof.

What is joint variation, and how is a joint variation problem solved?

Joint variation combines direct and inverse variation of several variables into one relationship, e.g. y varying directly as x and inversely as z is written y = kx/z. It is solved in three steps: write the proportionality, substitute one full set of known values to find k, then substitute the new values (with k now known) to find the requested unknown.

How does joint variation apply to a real-life law like Newton's law of gravitation?

The gravitational force G between two bodies varies directly as the product of their masses m1, m2 and inversely as the square of the distance d between them: $G \propto (m_1 \cdot m_2)/d^2$, i.e. $G = k(m_1 \cdot m_2)/d^2$. Given one complete set of G, m1, m2, d values, k can be found, after which G can be predicted for any new masses or distance.

Multiple Choice Questions (MCQs)

In a ratio a:b, a is called the:

A. Consequent

B. Antecedent

C. Mean

D. Extreme

Answer: B. Antecedent

Explanation: In a ratio a:b, a is the antecedent (the first term) and b is the consequent (the second term).

In the proportion a:b::c:d, the product of the extremes equals the product of the:

A. Antecedents

B. Means

C. Consequents

D. Ratios

Answer: B. Means

Explanation: The rule of proportion states ad = bc, i.e. the product of the extremes (a, d) equals the product of the means (b, c).

If y varies directly as x, this is written as:

A. $y = k/x$

B. $y = kx$

C. $xy = k$

D. $y = x/k$

Answer: B. $y = kx$

Explanation: Direct variation means $y \propto x$, i.e. $y = kx$ for a constant k.

If y varies inversely as x, this is written as:

A. $y = kx$

B. $xy = k$

C. $y - x = k$

D. $y + x = k$

Answer: B. $xy = k$

Explanation: Inverse variation means $y \propto 1/x$, i.e. $y = k/x$, equivalently $xy = k$.

The third proportional to a and b is given by:

A. b^2/a

B. a^2/b

C. ab

D. a/b

Answer: A. b^2/a

Explanation: For $a:b::b:c$, cross-multiplying gives $ac = b^2$, so the third proportional $c = b^2/a$.

The mean proportional between a and c is:

A. ac

B. $a+c$

C. $\pm\sqrt{(ac)}$

D. $(a+c)/2$

Answer: C. $\pm\sqrt{(ac)}$

Explanation: For $a:b::b:c$, $b^2 = ac$, so the mean proportional $b = \pm\sqrt{(ac)}$.

If a:b = c:d, then by invertendo:

A. $a:c = b:d$

B. $b:a = d:c$

C. $(a+b):b = (c+d):d$

D. $(a-b):b=(c-d):d$

Answer: B. $b:a = d:c$

Explanation: Invertendo simply flips both ratios: $a:b=c:d$ becomes $b:a=d:c$.

If a:b = c:d, then by componendo-dividendo:

A. $(a+b):(a-b) = (c+d):(c-d)$

B. $a:c = b:d$

C. $b:a = d:c$

D. $ab = cd$

Answer: A. (a+b):(a-b) = (c+d):(c-d)

Explanation: Componendo-dividendo combines componendo and dividendo to give (a+b):(a-b) = (c+d):(c-d).

In the k-method, if a/b = c/d = k, then a and c equal:

A. a=bk, c=dk

B. a=b/k, c=d/k

C. a=k/b, c=k/d

D. a=b+k, c=d+k

Answer: A. a=bk, c=dk

Explanation: Setting the common ratio equal to k gives a = bk and c = dk directly from a/b=k and c/d=k.

If y varies jointly as x and inversely as z, the relationship is written as:

A. y = kxz

B. y = kx/z

C. y = kz/x

D. y = k/(xz)

Answer: B. y = kx/z

Explanation: Joint variation combining direct variation with x and inverse variation with z gives y = kx/z.

Quick Revision Summary

- Ratio a:b compares two like quantities; no units; proportion a:b::c:d means ad=bc
- Direct variation: $y \propto x \Rightarrow y=kx$ | Inverse variation: $y \propto 1/x \Rightarrow y=k/x$ ($xy=k$)
- Third proportional to a,b: $c=b^2/a$ | Fourth proportional to a,b,c: $d=bc/a$
- Mean proportional between a,c: $b=\pm\sqrt{(ac)}$
- Invertendo: b:a=d:c | Alternendo: a:c=b:d
- Componendo: (a+b):b=(c+d):d | Dividendo: (a-b):b=(c-d):d
- Componendo-Dividendo: (a+b):(a-b)=(c+d):(c-d) — most useful for sum-over-difference equations
- K-method: set a/b=c/d=k ⇒ a=bk, c=dk, then substitute and simplify both sides
- Joint variation: $y \propto x/z \Rightarrow y=kx/z$; solve by finding k from one known set of values first

- Newton's gravitation, Hooke's law, kinetic energy are all joint-variation relationships
- Always check whether a solution makes physical sense in real-life variation problems

Exam Tips

- Whenever you see 'product of extremes = product of means', cross-multiply immediately — it solves most proportion problems in one step
- For variation word problems, first decide which quantities are direct (numerator) and which are inverse (denominator) before writing the k-equation
- Memorize componendo-dividendo specifically — it appears most often in exams and solves equations that already look like a sum over a difference
- When using the k-method, always substitute into BOTH sides separately, then simplify each side down before comparing them
- In joint variation, find k using the FIRST complete set of given values before touching the question being asked
- Double-check units and signs in real-life variation problems — reject any answer that doesn't make physical sense (e.g. a negative length)

Unit 4: Partial Fractions

A fraction is simply the quotient of two algebraic expressions, and when both the numerator and denominator are polynomials, the result is called a rational fraction. This short but important unit teaches the reverse of a familiar skill: instead of combining several simple fractions into one complicated fraction (as done when adding fractions with different denominators), partial fractions breaks a single complicated rational fraction back down into a sum of simpler ones -- a technique used constantly in calculus (integration), engineering (control systems), and higher algebra.

The unit first distinguishes proper fractions (numerator's degree less than denominator's) from improper fractions (numerator's degree greater than or equal to denominator's), showing that every improper fraction can be reduced by division to a polynomial plus a proper fraction. It then develops four separate rules for resolving a proper fraction into partial fractions, one for each type of factor that can appear in the denominator: non-repeated linear factors, repeated linear factors, non-repeated (irreducible) quadratic factors, and repeated quadratic factors.

Learning Objectives

- Define a fraction, a rational fraction, a proper fraction, and an improper fraction
- Convert an improper fraction into a polynomial plus a proper fraction using long division
- Resolve a fraction into partial fractions when the denominator has non-repeated linear factors
- Resolve a fraction into partial fractions when the denominator has repeated linear factors
- Resolve a fraction into partial fractions when the denominator has non-repeated irreducible quadratic factors
- Resolve a fraction into partial fractions when the denominator has repeated irreducible quadratic factors

Key Concepts

4.1 Fractions: Rational, Proper, and Improper

The quotient of two algebraic expressions is a fraction, written with the dividend above a bar and the divisor below it -- undefined wherever the divisor equals zero. When both the numerator N(x) and denominator D(x) are polynomials with real coefficients (and D(x) is not zero), the expression N(x)/D(x) is called a rational fraction, e.g. (x^2+3)/((x+1)^2(x+2)). A rational fraction is called proper if the degree of N(x) is LESS than the degree of D(x) (e.g. 2/(x+1)), and improper if the degree of N(x) is GREATER THAN OR EQUAL TO the degree of D(x) (e.g. 5x/(x+2)).

Every improper fraction can be reduced, by ordinary polynomial long division, to the sum of a quotient polynomial Q(x) and a proper fraction R(x)/D(x), where R(x) is the remainder and has a lower degree than D(x): N(x)/D(x) = Q(x) + R(x)/D(x). For example, dividing x^2+1 by x+1 gives quotient x-1 and remainder 2, so (x^2+1)/(x+1) = (x-1) + 2/(x+1). This step must always be done FIRST if a given fraction is improper -- partial fraction resolution itself only ever applies to a proper fraction.

4.2 Non-Repeated Linear Factors (Rule I)

If the denominator D(x) factors into distinct (non-repeated) linear factors, (a1x+b1)(a2x+b2)...(anx+bn), then the proper fraction N(x)/D(x) can be written as a sum of simple fractions, one per linear factor: A1/(a1x+b1) + A2/(a2x+b2) + ... + An/(anx+bn), where each Ai is a constant to be found. To find the constants, multiply both sides by the full denominator D(x) to clear all fractions, producing a polynomial identity that holds for every value of x.

The fastest way to find each constant is the 'zero method': substitute the value of x that makes ONE particular linear factor zero (e.g. x=4 for the factor x-4) into the cleared identity -- every term containing a different factor vanishes, leaving an equation with only that factor's constant, which can be solved immediately. Repeating this for each factor's root finds every constant one at a time, without ever needing to solve a system of equations.

4.3 Repeated Linear Factors (Rule II)

If a linear factor $(ax+b)$ occurs n times in the denominator (n is at least 2), it contributes NOT ONE but n separate partial fractions, with increasing powers of $(ax+b)$ in each denominator: $A_1/(ax+b) + A_2/(ax+b)^2 + \dots + A_n/(ax+b)^n$, where A_1 through A_n are constants to be found. This reflects the fact that a repeated root needs more 'degrees of freedom' in the partial fraction decomposition than a single occurrence would.

After multiplying through by the full denominator to clear fractions, the zero method still finds the constant attached to the HIGHEST power directly (substituting the repeated factor's root eliminates every other term). The remaining constants (for the lower powers of that same factor, and for any other distinct factors) are then found either by substituting other convenient values of x or by comparing coefficients of matching powers of x on both sides of the identity.

4.4 Non-Repeated Quadratic Factors (Rule III)

If an irreducible quadratic factor (ax^2+bx+c) -- one that cannot be factored further into real linear factors -- occurs ONCE in the denominator, its partial fraction takes the form $(Ax+B)/(ax^2+bx+c)$, with a LINEAR expression (not just a constant) in the numerator, since a quadratic denominator generally needs two independent constants (A and B) to match.

The solving procedure again starts by clearing all fractions through multiplication by the full denominator. Since a quadratic factor has no real root to substitute directly (the zero method doesn't fully apply to it), any real roots from OTHER linear factors present are substituted first to find their constants quickly, and the remaining unknowns (A and B for the quadratic term) are found by comparing coefficients of matching powers of x (e.g. x^2, x^1, x^0) on both sides of the cleared identity.

4.5 Repeated Quadratic Factors (Rule IV)

If an irreducible quadratic factor (ax^2+bx+c) occurs TWICE in the denominator, it contributes two partial fractions with increasing powers of the quadratic in the denominator, each with a linear numerator: $(Ax+B)/(ax^2+bx+c) + (Cx+D)/$

(ax^2+bx+c)^2, where A, B, C, D are four constants to be found -- combining the 'repeated factor needs more terms' idea from Rule II with the 'quadratic factor needs a linear numerator' idea from Rule III.

As with the other rules, the general method is: clear all fractions by multiplying through by the full denominator, substitute any real roots available from other linear factors to find their constants immediately, then compare coefficients of matching powers of x for the remaining unknowns. This four-rule system (combining repeated/non-repeated with linear/quadratic) covers every possible type of denominator factorization that can appear in this course.

Important Definitions

What is a rational fraction?

An expression N(x)/D(x) where N(x) and D(x) are polynomials with real coefficients and D(x) is not zero.

What is a proper fraction?

A rational fraction N(x)/D(x) where the degree of N(x) is less than the degree of D(x).

What is an improper fraction?

A rational fraction N(x)/D(x) where the degree of N(x) is greater than or equal to the degree of D(x).

How is an improper fraction converted to a proper fraction?

By polynomial long division: N(x)/D(x) = Q(x) + R(x)/D(x), where Q(x) is the quotient and R(x)/D(x) is a proper fraction (R(x) has lower degree than D(x)).

What partial fraction form corresponds to a non-repeated linear factor (ax+b)?

A single term A/(ax+b), where A is a constant to be found.

What partial fraction form corresponds to a linear factor (ax+b) repeated n times?

n separate terms: A1/(ax+b) + A2/(ax+b)^2 + ... + An/(ax+b)^n.

What partial fraction form corresponds to a non-repeated irreducible quadratic factor?

(Ax+B)/(ax^2+bx+c), with a linear expression Ax+B in the numerator.

What partial fraction form corresponds to an irreducible quadratic factor repeated twice?

$(Ax+B)/(ax^2+bx+c) + (Cx+D)/(ax^2+bx+c)^2$.

What is the 'zero method' for finding partial fraction constants?

Substituting the value of x that makes one particular linear factor zero, which eliminates every other term and isolates that factor's constant directly.

What is an identity in the context of partial fractions?

An equation that holds true for every value of the variable, such as the cleared equation obtained after multiplying both sides of a partial fraction decomposition by the full denominator.

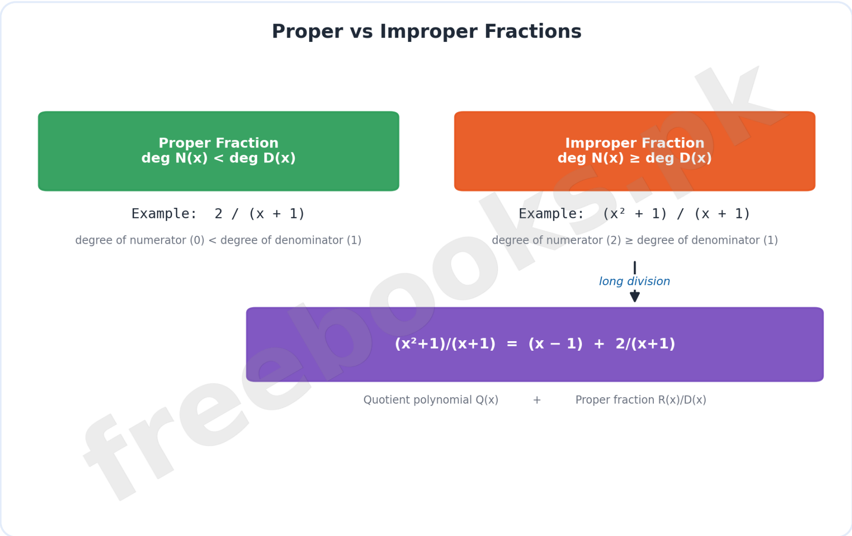
Key Facts and Relations

Topic	Key Fact / Relation
Proper fraction condition	degree $N(x) <$ degree $D(x)$
Improper fraction condition	degree $N(x) >=$ degree $D(x)$
Improper-to-proper reduction	$N(x)/D(x) = Q(x) + R(x)/D(x)$
Rule I (non-repeated linear)	$A/(ax+b)$
Rule II (linear repeated n times)	$A1/(ax+b) + A2/(ax+b)^2 + ... + An/(ax+b)^n$
Rule III (non-repeated quadratic)	$(Ax+B)/(ax^2+bx+c)$
Rule IV (quadratic repeated twice)	$(Ax+B)/(ax^2+bx+c) + (Cx+D)/(ax^2+bx+c)^2$
General method step 1	Ensure N(x) has lower degree than D(x) (divide first if not)
General method step 2	Multiply both sides by the full denominator (L.C.M.) to clear fractions

Topic	Key Fact / Relation
General method step 3	Use the zero method and/or compare coefficients of matching powers of x

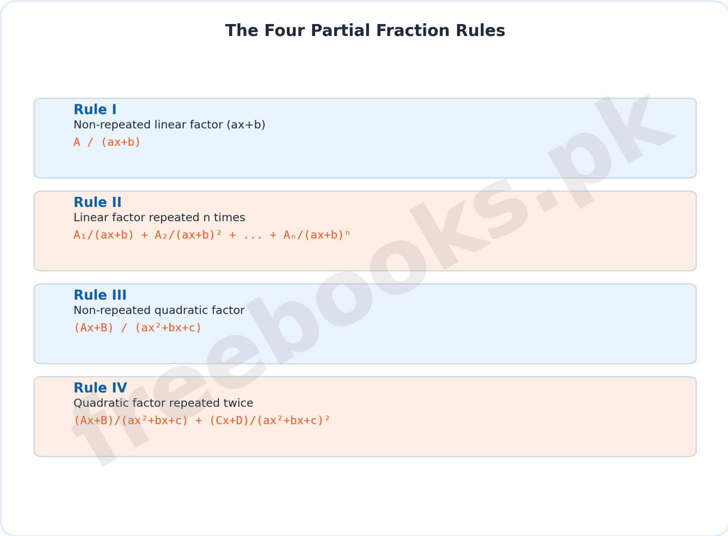
Diagrams

Proper vs Improper Fractions: A comparison diagram showing a proper fraction (numerator degree lower) alongside an improper fraction being reduced by long division into a quotient polynomial plus a proper fraction remainder



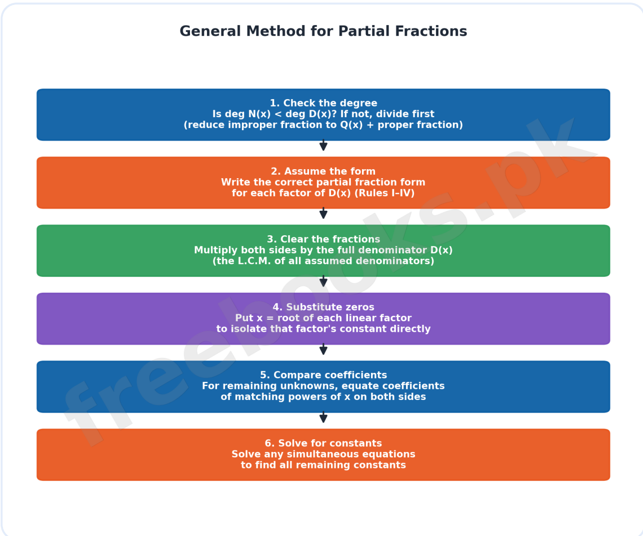
Proper vs Improper Fractions

The Four Partial Fraction Rules: A summary chart listing the four denominator factor types (non-repeated linear, repeated linear, non-repeated quadratic, repeated quadratic) alongside the partial fraction form each rule produces



The Four Partial Fraction Rules

General Method Flow: A flow diagram showing the step-by-step general method for resolving any rational fraction into partial fractions: check degree, multiply by L.C.M., substitute zeros, compare coefficients, solve for constants



General Method Flow

Short Questions & Answers

Define a proper fraction.

A rational fraction N(x)/D(x) where the degree of N(x) is less than the degree of D(x).

Define an improper fraction.

A rational fraction N(x)/D(x) where the degree of N(x) is greater than or equal to the degree of D(x).

What form does the partial fraction take for a non-repeated linear factor (ax+b)?

A/(ax+b), where A is a constant to be determined.

What form does the partial fraction take for a non-repeated irreducible quadratic factor?

(Ax+B)/(ax^2+bx+c), with a linear numerator Ax+B.

How many partial fraction terms does a linear factor repeated 3 times produce?

Three terms: A1/(ax+b) + A2/(ax+b)^2 + A3/(ax+b)^3.

What is the first step if the given fraction is improper?

Reduce it by long division into a quotient polynomial plus a proper fraction before applying any partial fraction rule.

Long Questions & Answers

Explain proper and improper fractions, and describe the general method used to resolve any rational fraction into partial fractions.

What is a rational fraction, and how is it classified as proper or improper?

A rational fraction is N(x)/D(x), where N(x) and D(x) are polynomials with real coefficients and D(x) is not zero. It is proper if the degree of N(x) is less than the degree of D(x) (e.g. 2/(x+1)), and improper if the degree of N(x) is greater than or equal to the degree of D(x) (e.g. 5x/(x+2)).

How is an improper fraction converted into a proper fraction?

By polynomial long division: dividing N(x) by D(x) gives a quotient polynomial Q(x) and a remainder R(x) of lower degree than D(x), so N(x)/D(x) = Q(x) + R(x)/D(x). For example, (x^2+1)/(x+1) = (x-1) + 2/(x+1), where x-1 is the quotient and 2/(x+1) is the resulting proper fraction.

What is the first step of the general method for resolving a fraction into partial fractions?

Confirm the fraction is proper (numerator's degree lower than the denominator's); if it is improper, divide first and only apply partial fractions to the resulting proper remainder fraction.

What are the middle steps of the general method?

Assume the correct partial fraction form based on the denominator's factors (linear/quadratic, repeated/non-repeated), then multiply both sides by the full denominator (the L.C.M. of all the assumed denominators) to clear every fraction, producing a polynomial identity true for all values of x.

How are the unknown constants finally found?

By substituting values of x that make individual linear factors zero (the 'zero method', which isolates one constant at a time), and/or by comparing the coefficients of matching powers of x on both sides of the cleared identity -- solving the resulting simultaneous equations for any constants not found directly by substitution.

State the four rules for resolving a fraction into partial fractions based on the type of factor in the denominator, with the partial fraction form each rule produces.

What is Rule I, for non-repeated linear factors?

If (ax+b) occurs once as a factor of D(x), it contributes a single partial fraction A/(ax+b), where A is a constant found (most easily) using the zero method: substituting the value of x that makes ax+b zero.

What is Rule II, for repeated linear factors?

If (ax+b) occurs n times as a factor of D(x) (n at least 2), it contributes n separate partial fractions with increasing powers in the denominator: A1/(ax+b) + A2/(ax+b)^2 + ... + An/(ax+b)^n, reflecting the extra degrees of freedom a repeated root requires.

What is Rule III, for non-repeated irreducible quadratic factors?

If an irreducible quadratic (ax^2+bx+c) occurs once in D(x), it contributes a single partial fraction $(Ax+B)/(ax^2+bx+c)$, with a LINEAR numerator (two constants A, B) since a quadratic denominator cannot be matched by a single constant alone.

What is Rule IV, for repeated irreducible quadratic factors?

If an irreducible quadratic occurs twice in D(x), it contributes two partial fractions, each with a linear numerator: $(Ax+B)/(ax^2+bx+c) + (Cx+D)/(ax^2+bx+c)^2$ -- combining the repeated-factor idea from Rule II with the linear-numerator idea from Rule III.

How are these four rules combined for a denominator with mixed factor types?

Each distinct factor of the denominator is handled by whichever rule matches its type (linear or quadratic, repeated or not), and the full partial fraction decomposition is simply the SUM of the term(s) each individual factor contributes -- the unknown constants across all terms are then found together by clearing fractions and comparing coefficients or substituting convenient values of x.

Multiple Choice Questions (MCQs)

A rational fraction N(x)/D(x) is called proper if:

A. degree N(x) > degree D(x)
B. degree N(x) < degree D(x)
C. degree N(x) = degree D(x)
D. D(x) = 0
Answer: B. degree N(x) < degree D(x)
Explanation: A proper fraction has the numerator's degree strictly less than the denominator's degree.

An improper fraction can always be reduced by division to:

A. Two proper fractions
B. A polynomial plus a proper fraction
C. A single constant

D. Another improper fraction

Answer: B. A polynomial plus a proper fraction

Explanation: Long division of N(x) by D(x) gives a quotient polynomial Q(x) and a proper remainder fraction R(x)/D(x).

For a non-repeated linear factor (ax+b) in the denominator, the partial fraction form is:

A. A/(ax+b)^2

B. (Ax+B)/(ax+b)

C. A/(ax+b)

D. A/(ax+b)^n

Answer: C. A/(ax+b)

Explanation: Rule I assigns a single constant-over-linear-factor term, A/(ax+b), to each non-repeated linear factor.

If a linear factor (ax+b) occurs 3 times in the denominator, how many partial fraction terms does it produce?

A. 1

B. 2

C. 3

D. 4

Answer: C. 3

Explanation: Rule II produces n separate terms (with increasing powers of the factor) for a factor repeated n times, so 3 terms for n=3.

For a non-repeated irreducible quadratic factor, the partial fraction numerator is:

A. A constant only

B. A linear expression Ax+B

C. A quadratic expression

D. Zero

Answer: B. A linear expression Ax+B

Explanation: Rule III requires a linear numerator (Ax+B) since a quadratic denominator generally needs two constants to be matched.

The 'zero method' for finding partial fraction constants works by:

A. Setting all constants to zero

B. Substituting a value of x that makes one linear factor zero

C. Multiplying by zero

D. Ignoring the denominator

Answer: B. Substituting a value of x that makes one linear factor zero

Explanation: Substituting the root of one linear factor makes every other term in the cleared identity vanish, isolating that factor's constant.

Before applying partial fraction rules, an improper fraction must first be:

A. Multiplied by its denominator

B. Reduced to a polynomial plus a proper fraction via division

C. Squared

D. Left unchanged

Answer: B. Reduced to a polynomial plus a proper fraction via division

Explanation: Partial fraction decomposition only applies directly to proper fractions, so an improper fraction must be divided down first.

For a quadratic factor repeated twice, the partial fraction form is:

A. (Ax+B)/(ax^2+bx+c) only

B. A/(ax^2+bx+c)^2 only

C. (Ax+B)/(ax^2+bx+c) + (Cx+D)/(ax^2+bx+c)^2

D. A + B + C + D

Answer: C. (Ax+B)/(ax^2+bx+c) + (Cx+D)/(ax^2+bx+c)^2

Explanation: Rule IV combines the repeated-factor and linear-numerator ideas, giving two linear-over-quadratic terms with increasing powers.

To clear all fractions when finding partial fraction constants, both sides of the equation are multiplied by:

A. The numerator N(x)

B. The full denominator D(x)

C. A single constant

D. Nothing needs to be multiplied

Answer: B. The full denominator D(x)

Explanation: Multiplying through by the full denominator D(x) clears every fraction, producing a polynomial identity true for all x.

An identity in algebra is an equation that:

A. Holds for only one value of x

B. Holds for all values of the variable

C. Has no solution

D. Is always false

Answer: B. Holds for all values of the variable

Explanation: An identity is satisfied by every value of the variable involved, unlike a conditional equation with specific solutions.

Quick Revision Summary

- Rational fraction N(x)/D(x): proper if deg N < deg D, improper if deg N >= deg D
- Improper fraction: N(x)/D(x) = Q(x) + R(x)/D(x) via long division
- Rule I (non-repeated linear factor ax+b): A/(ax+b)
- Rule II (linear factor repeated n times): A1/(ax+b) + A2/(ax+b)^2 + ... + An/(ax+b)^n
- Rule III (non-repeated quadratic factor): (Ax+B)/(ax^2+bx+c)
- Rule IV (quadratic factor repeated twice): (Ax+B)/(ax^2+bx+c) + (Cx+D)/(ax^2+bx+c)^2
- Zero method: substitute the root of one linear factor to isolate its constant directly
- For quadratic/unfound constants: compare coefficients of matching powers of x
- Always clear fractions first by multiplying both sides by the full denominator
- An identity holds for every value of x, not just specific solutions

Exam Tips

- Always check whether the fraction is proper or improper FIRST — partial fraction rules only apply to a proper fraction
- Use the zero method wherever a real linear factor exists — it's faster than comparing coefficients
- For quadratic factors with no real root, fall back to comparing coefficients of matching powers of x
- Write out the assumed partial fraction form completely before multiplying through — this avoids missing a term
- Double-check your final answer by adding the partial fractions back together and confirming you get the original fraction

- For repeated factors, remember EVERY power up to n needs its own term — a common mistake is only writing the highest power

Unit 5: Sets and Functions

This unit revisits sets -- well-defined collections of objects -- and builds up the formal machinery used throughout higher mathematics to combine and compare them: union, intersection, difference, and complement, followed by seven fundamental properties (commutative, associative, and distributive laws, plus De Morgan's laws) that these operations obey, each provable directly from the definitions using basic logic. Venn diagrams, introduced by John Venn, give a visual way to represent these operations and to verify the same properties geometrically by shading regions.

The unit then moves from sets to relations and functions: ordered pairs and the Cartesian product of two sets form the foundation for defining a binary relation as any subset of a Cartesian product, and a function as a special, more restrictive kind of relation where every element of the domain maps to exactly one element of the range. The unit closes by classifying functions into four key types -- into, one-one (injective), onto (surjective), and bijective (one-one and onto) -- which describe exactly how the elements of one set correspond to the elements of another.

Learning Objectives

- Recall the standard number sets N, W, Z, E, O, P, and Q, and recognize the set operations union, intersection, difference, and complement
- Perform union, intersection, difference, and complement operations on given sets
- State and prove the commutative, associative, and distributive properties of union and intersection, and De Morgan's laws
- Verify these fundamental properties for specific given sets by direct computation
- Use Venn diagrams to represent union, intersection, and complement, and to verify the fundamental properties
- Recognize ordered pairs and compute the Cartesian product of two sets
- Define a binary relation and identify its domain and range

- Define a function and identify its domain, co-domain, and range
- Distinguish into, one-one, onto, and bijective functions, and examine whether a given relation is a function
- Differentiate between a one-one function and a one-to-one correspondence

Key Concepts

5.1 Sets and Basic Operations

A set is a well-defined collection of objects, denoted by capital letters. Standard number sets used throughout mathematics include N (natural numbers, 1,2,3,...), W (whole numbers, 0,1,2,3,...), Z (integers), E and O (even and odd integers), P (prime numbers), Q (rational numbers, expressible as m/n with m,n integers and n not 0), Q' (irrational numbers), and R = Q union Q' (real numbers).

Four operations combine or compare sets. Union A union B is the set of elements in A, in B, or in both. Intersection A intersect B is the set of elements common to both A and B. Difference A - B (or A\B) is the set of elements in A but NOT in B. Complement A' (relative to a universal set U) is the set of elements in U but not in A, i.e. A' = U - A. Each is computed directly by listing which elements of the given sets satisfy the operation's defining condition.

5.2 Properties of Union and Intersection

Union and intersection obey several fundamental properties, each provable using the 'double subset' method: show every element of the left side is in the right side, and vice versa, so the two sets are equal. Commutative property: A union B = B union A, and A intersect B = B intersect A (order doesn't matter). Associative property: (A union B) union C = A union (B union C), and similarly for intersection (grouping doesn't matter).

Distributive property of union over intersection: A union (B intersect C) = (A union B) intersect (A union C). Distributive property of intersection over union: A intersect (B union C) = (A intersect B) union (A intersect C). De Morgan's laws connect complement with union/intersection: (A union B)' = A' intersect B', and (A intersect B)' = A' union B' -- the complement of a union is the intersection of the complements, and vice versa. All seven properties can also be VERIFIED (not just proved abstractly) for specific given sets by computing both sides directly and checking they match.

5.3 Venn Diagrams

A Venn diagram, introduced by British mathematician John Venn, represents a universal set U as a rectangle, with its subsets A, B, etc. drawn as closed figures (usually circles) inside it. Union, intersection, difference, and complement are all represented by shading the appropriate region: union shades everything in either circle, intersection shades only the overlapping region, and complement shades everything in the rectangle outside the given circle.

Venn diagrams also give a visual way to VERIFY the fundamental properties: draw the region for each side of a property (e.g. A union B using horizontal shading, B union A using vertical shading), and confirm the shaded regions are identical. This method is especially useful for checking De Morgan's laws and the distributive laws, where the algebraic proof involves several steps but the diagram makes the equality immediately visible.

5.4 Ordered Pairs and Cartesian Product

An ordered pair (x, y) is a pair of numbers where ORDER matters -- (x,y) is generally different from (y,x) unless x = y, and two ordered pairs (x,y) and (s,t) are equal only when x=s AND y=t simultaneously. This differs from a set, where {x,y} = {y,x} always.

The Cartesian product of two non-empty sets A and B, written A times B, is the set of ALL ordered pairs (x,y) where x is from A and y is from B. If A has m elements and B has n elements, then A times B has exactly m times n ordered pairs. In general A times B is NOT equal to B times A (unless A = B), since swapping the sets changes which set each coordinate comes from.

5.5 Binary Relations

A binary relation R from set A into set B is any subset of the Cartesian product A times B -- i.e., any collection of ordered pairs (x,y) with x from A and y from B, selected according to some rule or condition connecting x and y (such as y = 2x, or x+y = 6).

The domain of a relation R, written Dom R, is the set of all FIRST elements appearing in R's ordered pairs (a subset of A). The range of R, written Rang R, is the set of all SECOND elements appearing in R's ordered pairs (a subset of B). Both are found simply by listing the first and second coordinates of every ordered pair in the relation.

5.6 Functions: Domain, Co-domain, Range, and Types

A function f: A to B is a special kind of relation satisfying two conditions: Dom f = A (every element of A appears as a first coordinate), and every x in A appears in exactly ONE ordered pair of f (no element of A maps to two different outputs). Equivalently, every x in A is assigned a unique image y = f(x) in B. Here A is called the domain, B the co-domain, and the actual set of outputs achieved is the range (a subset of, or equal to, the co-domain).

Functions are classified into four key types based on how domain elements map to co-domain elements. An into function has range strictly smaller than the co-domain (some element of B is never used). A one-one (injective) function has every distinct input mapping to a distinct output (f(x1)=f(x2) forces x1=x2). An onto (surjective) function has range EQUAL to the co-domain (every element of B is used at least once). A bijective function (one-to-one correspondence) is both one-one AND onto simultaneously -- every element of A pairs with exactly one element of B and vice versa. Every function is a relation, but not every relation is a function; and a function need not be one-one or onto unless specifically shown to be.

Important Definitions

What is a set?

A well-defined collection of objects, denoted by a capital letter such as A, B, or C.

What is the union of two sets A and B?

The set of all elements that are in A, in B, or in both: $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$.

What is the intersection of two sets A and B?

The set of all elements common to both A and B: $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

What is the complement of a set A?

The set of elements in the universal set U that are not in A: A' = U - A.

What is an ordered pair?

A pair of numbers (x, y) in which the order matters, so (x,y) is generally different from (y,x) unless x = y.

What is the Cartesian product of sets A and B?

The set A × B of all ordered pairs (x,y) with x ∈ A and y ∈ B; if A has m elements and B has n elements, A × B has m×n ordered pairs.

What is a binary relation from set A to set B?

Any subset R of the Cartesian product A × B, consisting of ordered pairs connecting elements of A to elements of B.

What is a function f: A → B?

A relation where Dom f = A and every element of A is paired with exactly one (unique) element of B.

What is a one-one (injective) function?

A function where distinct elements of the domain always map to distinct elements of the co-domain: f(x1)=f(x2) implies x1=x2.

What is a bijective function?

A function that is both one-one and onto -- every element of A corresponds to exactly one element of B, and vice versa.

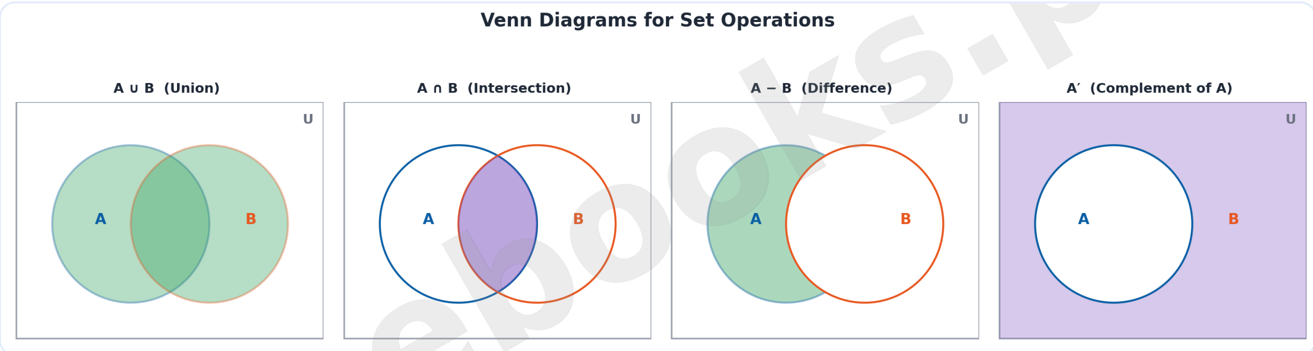
Key Facts and Relations

Topic	Key Fact / Relation
Union	A ∪ B = {x | x ∈ A or x ∈ B}
Intersection	A ∩ B = {x | x ∈ A and x ∈ B}
Difference	A - B = {x | x ∈ A and x ∉ B}
Complement	A' = U - A
Commutative laws	A∪B = B∪A, A∩B = B∩A
Associative laws	(A∪B)∪C = A∪(B∪C), (A∩B)∩C = A∩(B∩C)

Topic	Key Fact / Relation
Distributive laws	A∪(B∩C) = (A∪B)∩(A∪C), A∩(B∪C) = (A∩B)∪(A∩C)
De Morgan's laws	(A∪B)' = A'∩B', (A∩B)' = A'∪B'
Cartesian product size	|A × B| = |A| × |B|
Function condition	Dom f = A, and every x ∈ A has exactly one image f(x) ∈ B

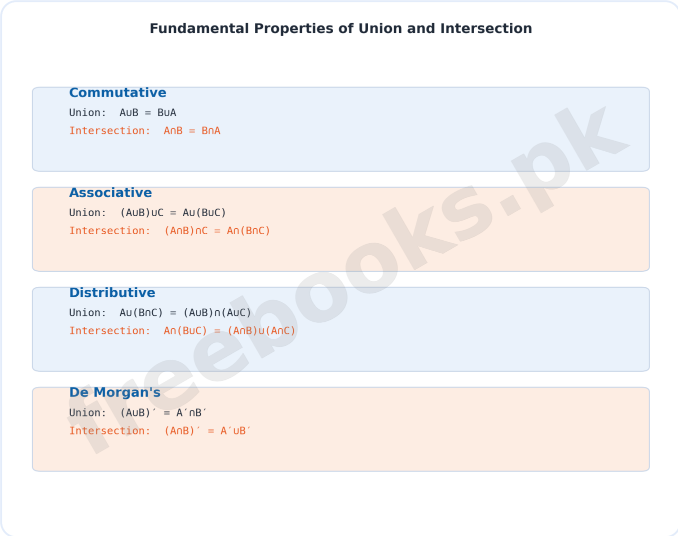
Diagrams

Venn Diagrams for Set Operations: A four-panel Venn diagram showing union, intersection, difference, and complement of two sets A and B inside a universal set U, with the relevant region shaded in each panel



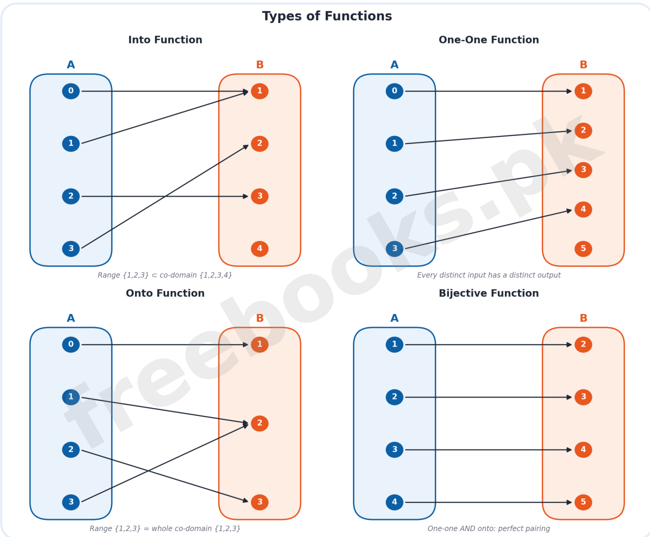
Venn Diagrams for Set Operations

Fundamental Properties Summary: A summary chart listing the commutative, associative, distributive, and De Morgan's properties of union and intersection side by side



Fundamental Properties Summary

Types of Functions: A comparison diagram of mapping arrows between two sets showing an into function, a one-one function, an onto function, and a bijective function



Types of Functions

Short Questions & Answers

Define the union of two sets.

$A \cup B$ is the set of all elements that are in A, in B, or in both.

Define the intersection of two sets.

$A \cap B$ is the set of all elements common to both A and B.

What is the complement of a set A relative to universal set U?

A' = U - A, the set of elements in U that are not in A.

State De Morgan's first law.

$(A \cup B)' = A' \cap B'$ -- the complement of a union equals the intersection of the complements.

What condition makes a relation a function?

Its domain equals the whole starting set A, and every element of A has exactly one image in B.

Define a bijective function.

A function that is both one-one and onto -- every element of A corresponds to exactly one element of B, and vice versa.

Long Questions & Answers

State the fundamental properties of union and intersection of sets (commutative, associative, distributive, De Morgan's), and explain how they are proved and verified.

What are the commutative and associative properties of union and intersection?

Commutative: $A \cup B = B \cup A$ and $A \cap B = B \cap A$ (order doesn't matter). Associative: $(A \cup B) \cup C = A \cup (B \cup C)$ and $(A \cap B) \cap C = A \cap (B \cap C)$ (grouping doesn't matter). Both are proved by showing an arbitrary element of the left side is always in the right side, and vice versa (the 'double subset' method).

What are the distributive properties connecting union and intersection?

Union distributes over intersection: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. Intersection distributes over union: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Each is proved the same double-subset way, breaking 'x is in A or (in B and in C)' into the equivalent 'x is in (A or B) and (A or C)' using basic logic.

What are De Morgan's laws?

$(A \cup B)' = A' \cap B'$ (the complement of a union is the intersection of the complements) and $(A \cap B)' = A' \cup B'$ (the complement of an intersection is the union of the complements). They are proved by translating 'x is not in A∪B' into 'x is not in A AND not in B', which is exactly the definition of x being in A'∩B'.

How is a property PROVED versus VERIFIED?

A proof shows the property holds for ANY sets A, B, C in general, using the double-subset method and the definitions of the operations. Verification instead picks SPECIFIC given sets, computes both sides of the property directly by listing elements, and confirms the two resulting sets are identical -- a proof establishes the general law, verification just checks one example obeys it.

How does a Venn diagram help verify these properties?

Each side of a property is drawn as a shaded region inside the universal-set rectangle (e.g. using horizontal lines for one side, vertical lines for the other); if the two shaded regions cover exactly the same area, the property is visually confirmed for any sets in that general configuration, without needing to list elements algebraically.

Explain ordered pairs, the Cartesian product, binary relations, and how a function is defined and classified into different types.

What is an ordered pair, and how does it differ from a set?

An ordered pair (x,y) is written with order mattering, so (x,y) generally differs from (y,x) unless x=y, and (x,y)=(s,t) only when x=s and y=t together. This differs from a set {x,y}, where the order of listing elements never matters.

What is the Cartesian product of two sets?

For non-empty sets A and B, A×B is the set of all ordered pairs (x,y) with x from A and y from B. If A has m elements and B has n elements, A×B has m×n ordered pairs, and in general A×B is not equal to B×A.

What is a binary relation, and what are its domain and range?

A binary relation R from A to B is any subset of A×B. Its domain, Dom R, is the set of all first elements of its ordered pairs; its range, Rang R, is the set of all second elements -- both found by listing the first and second coordinates of every pair in R.

What conditions must a relation satisfy to be a function?

A relation $f:A \rightarrow B$ is a function only if its domain equals the entire set A (every element of A is used), AND every element of A appears in exactly one ordered pair of f (no element of A has two different images) -- equivalently, every x in A has a unique image $y=f(x)$ in B.

What are the four key types of functions?

An into function has range strictly smaller than the co-domain. A one-one (injective) function sends distinct inputs to distinct outputs. An onto (surjective) function has range equal to the whole co-domain. A bijective function (one-to-one correspondence) is both one-one and onto at once, pairing every element of A with exactly one element of B and vice versa.

Multiple Choice Questions (MCQs)

$A \cup B$ is the set of elements that are:

A. Only in A
B. Only in B
C. In A, in B, or in both
D. In neither A nor B
Answer: C. In A, in B, or in both
Explanation: Union collects every element that belongs to A, to B, or to both sets.

$A \cap B$ is the set of elements that are:

A. In A or B
B. Common to both A and B
C. In A but not B
D. In neither set
Answer: B. Common to both A and B
Explanation: Intersection consists only of the elements shared by both A and B.

The complement A' of a set A (relative to U) is defined as:

A. A ∪ U

B. A ∩ U

C. U - A

D. A - U

Answer: C. U - A

Explanation: A' is everything in the universal set U that is not in A, i.e. U - A.

Which property states A ∪ B = B ∪ A?

A. Associative

B. Distributive

C. Commutative

D. De Morgan's

Answer: C. Commutative

Explanation: The commutative property says the order of union (or intersection) doesn't affect the result.

De Morgan's law states that (A ∪ B)' equals:

A. A' ∪ B'

B. A' ∩ B'

C. A ∩ B

D. A ∪ B

Answer: B. A' ∩ B'

Explanation: By De Morgan's law, the complement of a union is the intersection of the individual complements.

An ordered pair (x, y) is equal to (s, t) only when:

A. x = t and y = s

B. x = s and y = t

C. x + y = s + t

D. Never equal

Answer: B. x = s and y = t

Explanation: Two ordered pairs are equal only when their corresponding first and second coordinates both match.

If set A has 4 elements and set B has 3 elements, then A × B has:

A. 7 elements

B. 12 elements

C. 4 elements

D. 3 elements

Answer: B. 12 elements

Explanation: The Cartesian product A×B has |A|×|B| ordered pairs, so 4×3 = 12.

A binary relation from A to B is defined as:

A. Any element of A

B. Any subset of A × B

C. Any subset of A

D. A function only

Answer: B. Any subset of A × B

Explanation: A binary relation is any subset of the Cartesian product A×B, connecting elements of A to elements of B.

A function f: A → B must satisfy:

A. Dom f = B only

B. Every element of A has exactly one image in B

C. Range f = B always

D. A and B must be equal

Answer: B. Every element of A has exactly one image in B

Explanation: A function requires the domain to equal A and every element of A to map to exactly one unique image in B.

A function that is both one-one and onto is called:

A. Into

B. Injective only

C. Bijective

D. Not a function

Answer: C. Bijective

Explanation: A bijective function (one-to-one correspondence) is one that is simultaneously one-one and onto.

Quick Revision Summary

- Union A∪B: elements in A or B or both | Intersection A∩B: elements common to both
- Difference A-B: in A but not B | Complement A': U - A

- Commutative: $A \cup B = B \cup A$, $A \cap B = B \cap A$ | Associative: grouping doesn't matter
- Distributive: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- De Morgan's: $(A \cup B)' = A' \cap B'$, $(A \cap B)' = A' \cup B'$
- Venn diagrams: shade regions to represent AND verify set operations/ properties
- Ordered pair (x,y): order matters, unlike a set {x,y}
- Cartesian product $A \times B$: all ordered pairs (x,y), $x \in A$, $y \in B$; size = $|A| \times |B|$
- Binary relation: any subset of $A \times B$; Dom R = first elements, Rang R = second elements
- Function: Dom f = A, and every $x \in A$ has exactly ONE image in B
- Function types: into (range ⊂ co-domain), one-one (distinct→distinct), onto (range=co-domain), bijective (both)

Exam Tips

- When computing set operations, always list the elements clearly first — mistakes usually come from rushing the listing step
- Remember A - B and B - A are generally different — subtraction order matters just like with ordered pairs
- For proofs, always use the double-subset method: show left ⊆ right, then right ⊆ left, then conclude equality
- When verifying with Venn diagrams, use different shading styles (horizontal, vertical, crossed) for each side of the property being checked
- To check if a relation is a function, verify BOTH conditions: every domain element is used, AND no domain element repeats with a different output
- Memorize the four function types together as a group — into/onto describe the RANGE, one-one/bijective describe how inputs MAP to outputs

Unit 6: Basic Statistics

This unit introduces statistics as the branch of mathematics concerned with organizing raw data into a usable form. It begins with the frequency distribution -- a tabular arrangement that groups observations and counts how many fall into each group -- and the graphical tools built on top of it: the histogram (adjacent rectangles showing frequency per class), the frequency polygon (a line graph through class midpoints), and the cumulative frequency polygon or ogive (a running-total curve used later to read off the median and quartiles directly from a graph).

The unit then covers the two main families of statistical summary. Measures of central tendency -- arithmetic mean, median, mode, geometric mean, harmonic mean, and weighted mean -- each locate a single representative value around which the data tends to cluster, computed differently for ungrouped data versus grouped (frequency-table) data, by direct or indirect (assumed-mean) methods. Measures of dispersion -- range, variance, and standard deviation -- describe how spread out the data is around that central value, which is essential for comparing two data sets that may share the same mean but differ greatly in consistency.

Learning Objectives

- Construct a discrete and a continuous grouped frequency table from raw data
- Identify class limits, class boundaries, class marks (midpoints), and cumulative frequency for a given frequency table
- Construct a histogram for equal and unequal class intervals
- Construct a frequency polygon and a cumulative frequency polygon (ogive)
- Calculate the arithmetic mean for ungrouped and grouped data using direct and indirect (assumed mean) methods
- Calculate the median and mode for ungrouped and grouped (discrete and continuous) data
- Calculate the geometric mean, harmonic mean, and weighted mean, and compute moving averages
- Recognize and apply the properties of the arithmetic mean

- Estimate the median, quartiles, and mode graphically from an ogive and a histogram
- Calculate the range, variance, and standard deviation for ungrouped and grouped data

Key Concepts

6.1 Frequency Distribution and Its Construction

A frequency distribution is a tabular arrangement that classifies data into groups and records the number of observations (the frequency) falling into each group; data presented this way is called grouped data. A discrete frequency table is built by listing each distinct value of the variable and tallying how often it occurs. A continuous frequency table is built in steps: find the Range (Xmax − Xmin), decide the number of classes k (usually 5 to 20, depending on how large the range is), find the class size h = Range/k (rounding h up, e.g. treating 7.1 or 7.9 as 8 -- this relaxed rounding is called the rule of approximation), then mark off classes starting from the minimum observation and tally each value into its class.

Class limits are the stated minimum and maximum values of a class (e.g. 10 and 19 in the class 10-19). Because real measurements are continuous, class boundaries are used instead for calculation: a class boundary is found by adding two successive class limits and dividing by 2 (so the boundaries of 10-19 and 20-29 both use 19.5, the shared boundary between them). The class mark or midpoint of a class is the average of its lower and upper class limits, used to represent every observation in that class during calculations. Cumulative frequency is the running total of frequencies up to and including a given class's upper boundary.

6.2 Histograms and the Frequency Polygon

A histogram is a graph of adjacent rectangles built on the XY-plane, with class boundaries marked on the x-axis and frequency on the y-axis; each rectangle's height corresponds to its class's frequency. When all class intervals are equal in size this height is simply the frequency itself. When class intervals are unequal, the raw frequency cannot be used directly -- each class's frequency must first be divided by its own class interval size to get a proportional height, so that the

AREA of each bar (not just its height) stays proportional to its frequency, keeping the histogram visually honest.

A frequency polygon is a many-sided closed figure formed by plotting a point at each class's midpoint (class mark) against its frequency, then joining consecutive points with straight line segments. To close the figure at both ends, one extra class of the same size is added before the first class and after the last class, each assigned a frequency of zero, so the polygon returns to the x-axis on both sides rather than floating above it.

6.3 Cumulative Frequency Distribution and the Ogive

A cumulative frequency distribution (also called a 'less than' cumulative frequency distribution) is a table showing, against each class's upper class boundary, the running total of all frequencies up to that point. It is built by successively adding each class's frequency to the cumulative total of the classes before it.

The cumulative frequency polygon, or ogive, is the graph of this cumulative distribution: class boundaries are marked on the x-axis, cumulative frequency on the y-axis, the cumulative frequency at each upper class boundary is plotted as a point, consecutive points are joined by line segments, and a perpendicular is dropped from the final point to the x-axis to close the figure. The ogive's main practical use, covered later in the unit, is reading off the median and quartiles directly from the graph.

6.4 Measures of Central Tendency: Mean, Median, and Mode

The arithmetic mean ($\bar{X}$) is the sum of all observations divided by their number: $\bar{X} = \Sigma X/n$ for ungrouped data, or $\bar{X} = \Sigma fX/\Sigma f$ for grouped data (using each class's midpoint X). The indirect (short-cut/coding) method assumes a provisional value A, computes deviations $D = X - A$ (or coded deviations $u = D/h$), and recovers the true mean as $\bar{X} = A + \Sigma fD/\Sigma f$, or $\bar{X} = A + (\Sigma fu/\Sigma f)\times h$ -- useful for simplifying arithmetic with large or awkward values. Four key properties of the mean: the mean of a constant data set is that constant itself; the mean shifts by the same amount as a change of origin (adding a constant to every value adds that constant to the mean); the mean scales the same way as a change of scale (multiplying every value by a constant multiplies the mean by that constant);

and the sum of deviations of all observations from their own mean is always exactly zero.

The median is the middle-most observation once data is arranged in order, dividing the data into two equal halves. For ungrouped data with an odd number of observations n, the median is the value at position $(n+1)/2$; for an even n, it is the average of the values at positions $n/2$ and $(n/2)+1$. For grouped discrete data, the median is the value whose cumulative frequency first reaches or passes $(n/2)$. For grouped continuous data, the median class is the class containing the $(n/2)$th observation (found via cumulative frequency), and the exact median is $\text{Median} = l + (h/f)(n/2 - c)$, where l is the median class's lower class boundary, h its class size, f its frequency, and c the cumulative frequency of the class immediately before it. The mode is the most frequently occurring observation -- for ungrouped or discrete grouped data it is simply read off directly, while for continuous grouped data the modal class (highest frequency) is located first and then $\text{Mode} = l + [(f_m - f_1)/(2f_m - f_1 - f_2)]\times h$, where f_m is the modal class's frequency and f_1, f_2 are the frequencies of the classes immediately before and after it.

6.5 Geometric Mean, Harmonic Mean, Weighted Mean, and Moving Averages

The geometric mean (G.M.) of n observations is the nth positive root of their product: $\text{G.M.} = (x_1 \times x_2 \times \ldots \times x_n)^{(1/n)}$, most conveniently computed using logarithms as $\text{G.M.} = \text{Antilog}(\Sigma \log X / n)$ for ungrouped data or $\text{Antilog}(\Sigma f \log X / \Sigma f)$ for grouped data. The harmonic mean (H.M.) is the reciprocal of the mean of the reciprocals of the observations: $\text{H.M.} = n / \Sigma(1/X)$ for ungrouped data, or $\text{H.M.} = \Sigma f / \Sigma(f/X)$ for grouped data -- both are alternative averages used in specific situations (e.g. rates, ratios) where the simple arithmetic mean would be misleading.

The weighted arithmetic mean applies when observations are not all equally important: each value x_i is given a weight w_i reflecting its relative importance, and the weighted mean is $\bar{X}_w = \Sigma wx/\Sigma w$ (compare this to the ordinary mean, which implicitly treats every observation as equally weighted). Moving averages are successive arithmetic means computed over a fixed-size sliding window of consecutive time periods (e.g. a 3-day moving average): the average of the first window is placed at its midpoint, then the window slides forward by dropping the earliest period and adding the next one, repeating until the data is exhausted -- this smooths out short-term fluctuations and reveals the underlying trend.

6.6 Graphical Estimation and Measures of Dispersion

The median and quartiles can be located graphically from an ogive: to find the first quartile Q1, locate (n/4) on the y-axis, draw a horizontal line to where it meets the ogive curve, then drop a vertical line down to the x-axis and read the value. The same process applied at 2(n/4) gives the median (Q2), and at 3(n/4) gives the third quartile (Q3). The mode can be located graphically from a histogram: find the tallest rectangle (the modal class), draw a line from its top-left corner to the top-left corner of the next rectangle, another line from its top-right corner to the top-right corner of the previous rectangle, then drop a perpendicular from where these two lines cross down to the x-axis -- the point where it meets the x-axis is the mode.

Measures of dispersion describe how spread out or scattered the data is, which is essential because two data sets can share the same mean yet differ greatly in consistency. Range is the simplest measure: Range = $X_{max} - X_{min}$ for raw data, or (upper class boundary of the last class) − (lower class boundary of the first class) for grouped data. Variance is the mean of the squared deviations from the mean: $S^2 = \Sigma(X-\bar{X})^2/n$ for ungrouped data, or $S^2 = \Sigma fX^2/\Sigma f - (\Sigma fX/\Sigma f)^2$ for grouped data (a computationally faster equivalent form). Standard deviation is simply the positive square root of variance, $S = \sqrt{S^2}$ -- unlike variance, it is expressed in the same units as the original data, which is why it is the more commonly quoted measure of spread.

Important Definitions

What is a frequency distribution?

A tabular arrangement that classifies data into groups and records the number of observations (frequency) falling in each group; such data is called grouped data.

What are class boundaries?

The real limits of a class, found by adding two successive class limits and dividing by 2; the result is the upper boundary of one class and the lower boundary of the next.

What is the class mark (midpoint) of a class?

The average of a class's lower and upper class limits, used to represent every observation within that class in calculations.

What is cumulative frequency?

The running total of frequencies up to and including a given class's upper class boundary.

What is a histogram?

A graph of adjacent rectangles on the XY-plane, built on class boundaries, whose heights (or, for unequal intervals, proportional heights) represent frequency.

What is a frequency polygon?

A many-sided closed figure formed by plotting frequency against class midpoints and joining consecutive points with line segments, closed at both ends with zero-frequency classes.

What is an ogive?

A cumulative frequency polygon: a graph of the 'less than' cumulative frequency distribution, plotted against upper class boundaries.

What is the arithmetic mean?

The sum of all observations divided by their number: $\bar{X} = \Sigma X/n$ for ungrouped data, or $\bar{X} = \Sigma fX/\Sigma f$ for grouped data.

What is the median?

The middle-most observation in data arranged in order of magnitude, dividing the data set into two equal halves.

What is the mode?

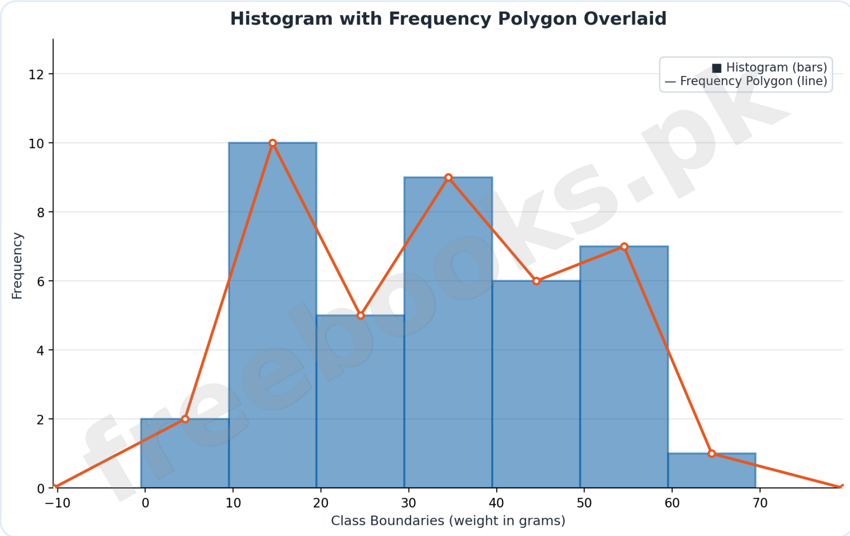
The value that occurs with the greatest frequency in a data set -- the single most commonly occurring observation.

Key Facts and Relations

Topic	Key Fact / Relation
Class size	$h = \text{Range} / k$ (k = number of classes)
Class boundary	average of two successive class limits
Class mark (midpoint)	$X = (\text{lower limit} + \text{upper limit}) / 2$
Arithmetic mean, direct method (grouped)	$\bar{X} = \Sigma fX / \Sigma f$
Arithmetic mean, indirect method (grouped)	$\bar{X} = A + (\Sigma fu / \Sigma f) \times h$
Median, grouped continuous data	$\text{Median} = l + (h/f)(n/2 - c)$
Mode, grouped continuous data	$\text{Mode} = l + [(f_m - f_1)/(2f_m - f_1 - f_2)] \times h$
Geometric mean, grouped data	$\text{G.M.} = \text{Antilog}(\Sigma f \log X / \Sigma f)$
Harmonic mean, grouped data	$\text{H.M.} = \Sigma f / \Sigma(f/X)$
Weighted mean	$\bar{X}_w = \Sigma wx / \Sigma w$
Range	$\text{Range} = X_{max} - X_{min}$
Variance and standard deviation, grouped data	$S^2 = \Sigma fX^2/\Sigma f - (\Sigma fX/\Sigma f)^2$, $S = \sqrt{S^2}$

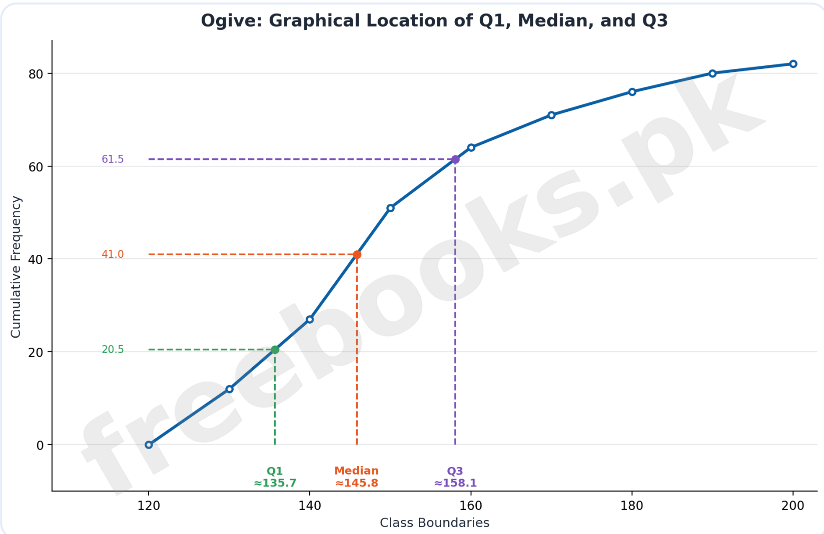
Diagrams

Histogram with Frequency Polygon: A histogram of adjacent rectangles for a grouped weight distribution, with a frequency polygon line overlaid through the midpoints of each class, closed at both ends with zero-frequency classes



Histogram with Frequency Polygon

Ogive with Graphical Quartiles: A cumulative frequency polygon (ogive) with dashed guide lines showing how the first quartile, median, and third quartile are read off graphically at n/4, 2n/4, and 3n/4 on the y-axis



Ogive with Graphical Quartiles

Central Tendency and Dispersion Summary: A summary chart listing the formulas for arithmetic mean, median, mode, and range/standard deviation, for both ungrouped and grouped data

Central Tendency and Dispersion Summary

Short Questions & Answers

Define a frequency distribution.

A frequency distribution is a tabular arrangement that classifies data into groups and records how many observations fall into each group.

What is the difference between class limits and class boundaries?

Class limits are the stated minimum and maximum values of a class, while class boundaries are the real limits obtained by averaging two successive class limits, used for accurate calculation with continuous data.

Define a histogram.

A histogram is a graph of adjacent rectangles built on class boundaries, whose heights represent the frequency (or proportional frequency, for unequal class intervals) of each class.

What is an ogive?

An ogive is a cumulative frequency polygon -- a graph of the 'less than' cumulative frequency distribution plotted against upper class boundaries.

Define mode for ungrouped data.

For ungrouped data, the mode is simply the observation that occurs most frequently in the data set.

What does standard deviation measure?

Standard deviation measures the typical spread of data around its mean, expressed in the same units as the original data; it is the positive square root of the variance.

Long Questions & Answers

Explain how a continuous frequency distribution is constructed, and describe how a histogram and a frequency polygon are drawn from it.

What are the steps in constructing a continuous frequency table?

Find the Range (Xmax − Xmin), choose the number of classes k (usually 5-20, depending on the range), compute the class size h = Range/k (rounding up per the rule of approximation), mark off classes starting from the minimum observation, and tally each observation into its class using tally marks.

What are class limits, class boundaries, and class marks?

Class limits are a class's stated minimum and maximum values; class boundaries are the real limits found by averaging two successive class limits; the class mark (midpoint) is the average of a class's lower and upper limits, used to represent that class in calculations.

How is a histogram constructed for equal class intervals, and how does it differ for unequal intervals?

For equal intervals, class boundaries go on the x-axis and each rectangle's height equals its frequency directly. For unequal intervals, each frequency must first be divided by its own class interval size to get a proportional height, so the bar's AREA (not just height) stays proportional to frequency.

How is a frequency polygon constructed?

Plot each class's midpoint against its frequency, add one zero-frequency class before the first and after the last class to close the figure at both ends, then join all the plotted points with straight line segments.

What is a cumulative frequency table and an ogive?

A cumulative frequency table shows the running total of frequencies up to each class's upper boundary; an ogive is the graph of this table, formed by plotting cumulative frequency against upper class boundaries and joining the points, used later to read off the median and quartiles graphically.

Explain how the mean, median, and mode are calculated for grouped data, and how the measures of dispersion describe the spread of the data.

How is the arithmetic mean calculated for grouped data using the direct and indirect methods?

The direct method uses $\bar{X} = \Sigma fX/\Sigma f$, where X is each class's midpoint. The indirect method assumes a provisional mean A, computes coded deviations $u = (X-A)/h$, and recovers the true mean as $\bar{X} = A + (\Sigma fu/\Sigma f)\times h$ -- useful for simplifying arithmetic with large or awkward class marks.

How is the median located and calculated for grouped continuous data?

First locate the median class -- the class whose cumulative frequency first reaches (n/2) -- then apply $\text{Median} = l + (h/f)(n/2 - c)$, where l is the median class's lower boundary, h its size, f its frequency, and c the cumulative frequency of the class before it.

How is the mode calculated for grouped continuous data?

Locate the modal class (the class with the highest frequency fm), note the frequencies f1 and f2 of the classes immediately before and after it, then apply $\text{Mode} = l + [(fm-f1)/(2fm-f1-f2)]\times h$, where l is the modal class's lower boundary and h its size.

What is the range, and how is it defined for grouped data?

Range is the simplest measure of dispersion: for raw data it is $X_{max} - X_{min}$; for grouped data it is the upper class boundary of the last class minus the lower class boundary of the first class.

What do variance and standard deviation measure, and how are they calculated?

Variance is the mean of the squared deviations of observations from their mean, $S^2 = \Sigma(X-\bar{X})^2/n$ (or the computational form $\Sigma fX^2/\Sigma f - (\Sigma fX/\Sigma f)^2$ for grouped data); standard deviation $S = \sqrt{S^2}$ is its square root, expressed in the original data's units, and is used to compare how consistent or spread out two data sets are even when they share the same mean.

Multiple Choice Questions (MCQs)

A tabular arrangement that classifies data into groups with their frequencies is called:

A. A tally chart
B. A frequency distribution
C. A class limit
D. An ogive
Answer: B. A frequency distribution
Explanation: This tabular arrangement, classifying data into groups and recording each group's frequency, is called a frequency distribution.

The size of a class interval h is calculated as:

A. k / Range
B. Range × k
C. Range / k
D. Range + k
Answer: C. Range / k
Explanation: The class size h is found by dividing the Range by the number of classes k: h = Range/k.

A class boundary is obtained by:

A. Multiplying two class limits
B. Averaging two successive class limits
C. Subtracting the class limits
D. Adding all class limits together
Answer: B. Averaging two successive class limits

Explanation: A class boundary is found by adding two successive class limits and dividing the sum by 2.

The class mark (midpoint) of the class 20 — 29 is:

A. 20

B. 24.5

C. 25

D. 29

Answer: B. 24.5

Explanation: The class mark is the average of the lower and upper limits: (20+29)/2 = 24.5.

A graph of adjacent rectangles built on class boundaries, with height representing frequency, is called:

A. A frequency polygon

B. An ogive

C. A histogram

D. A pie chart

Answer: C. A histogram

Explanation: This is the definition of a histogram -- adjacent rectangles on class boundaries, with height showing frequency.

A frequency polygon is formed by plotting frequency against:

A. Class limits

B. Class boundaries

C. Class marks (midpoints)

D. Cumulative frequency

Answer: C. Class marks (midpoints)

Explanation: A frequency polygon plots each class's frequency against its class mark (midpoint), then joins the points.

An ogive is another name for:

A. A histogram

B. A cumulative frequency polygon

C. A frequency table

D. A bar chart

Answer: B. A cumulative frequency polygon

Explanation: An ogive is the graph of the cumulative ('less than') frequency distribution, also called a cumulative frequency polygon.

For grouped continuous data, the median is calculated using the formula:

A. $l + (h/f)(n/2 - c)$

B. $l + [(f_m - f_1)/(2f_m - f_1 - f_2)] \times h$

C. $\Sigma fX/\Sigma f$

D. $X_{max} - X_{min}$

Answer: A. $l + (h/f)(n/2 - c)$

Explanation: The median for grouped continuous data uses Median = $l + (h/f)(n/2 - c)$, based on the median class's lower boundary, size, frequency, and preceding cumulative frequency.

The mode of a data set is defined as:

A. Its middle value

B. Its average value

C. Its most frequently occurring value

D. Its largest value

Answer: C. Its most frequently occurring value

Explanation: The mode is simply the value that occurs with the greatest frequency in the data set.

Standard deviation is defined as:

A. The mean of the data

B. The square of the variance

C. The positive square root of the variance

D. The range divided by 2

Answer: C. The positive square root of the variance

Explanation: Standard deviation is the positive square root of the variance, expressed in the same units as the original data.

Quick Revision Summary

- Frequency distribution: tabular grouping of data with frequency per group | grouped data = data in this form

- Class size $h = \text{Range}/k$ | Class boundary = average of two successive class limits | Class mark = (lower+upper limit)/2
- Cumulative frequency: running total of frequencies up to a class's upper boundary
- Histogram: adjacent rectangles on class boundaries, height = frequency (or proportional height for unequal intervals)
- Frequency polygon: line through class-mark/frequency points, closed with zero-frequency classes at both ends
- Ogive: graph of the cumulative frequency distribution; used to read median/quartiles graphically
- Arithmetic mean: $\bar{X} = \Sigma fX/\Sigma f$ (direct) or $A + (\Sigma fu/\Sigma f)\times h$ (indirect/short-cut)
- Mean properties: constant data → mean = constant | shifts with origin change | scales with scale change | $\Sigma(X-\bar{X}) = 0$
- Median (grouped continuous) = $l + (h/f)(n/2 - c)$ | Mode (grouped continuous) = $l + [(f_m-f_1)/(2f_m-f_1-f_2)]\times h$
- Geometric mean = $\text{Antilog}(\Sigma f \log X/\Sigma f)$ | Harmonic mean = $\Sigma f/\Sigma(f/X)$ | Weighted mean = $\Sigma wx/\Sigma w$
- Moving average: successive means over a sliding window of fixed size, smooths short-term fluctuation
- Range = $X_{max}-X_{min}$ | Variance $S^2 = \Sigma fX^2/\Sigma f - (\Sigma fX/\Sigma f)^2$ | Standard deviation $S = \sqrt{S^2}$

Exam Tips

- Always compute class boundaries before finding class marks or plotting a histogram — mixing up limits and boundaries is the most common calculation error in this unit
- For unequal-interval histograms, remember to divide frequency by class interval size first — skipping this step distorts the shape of the whole histogram
- When finding the median or mode for grouped data, locate the correct class FIRST using cumulative frequency, then plug values into the formula — never guess the class

- Use the indirect (assumed mean) method whenever raw values are large or awkward — it gives the exact same answer as the direct method with far less arithmetic
- On an ogive, always work in the same order: locate n/4, 2n/4, 3n/4 on the y-axis, draw horizontal then vertical guide lines, and read the x-axis value
- Remember variance is in squared units of the data (e.g. sq. gm) while standard deviation is back in the original units — always take the square root before comparing spread across two data sets

Unit 7: Introduction to Trigonometry

This unit opens trigonometry by first fixing how an angle is measured. The everyday sexagesimal system (degrees, minutes, and seconds, with 60'' = 1' and 60' = 1 degree) is joined by the circular (radian) system used throughout higher mathematics, where one radian is the angle subtended by an arc equal in length to the radius; the two systems are linked by the identity 180 degrees = pi radians. Once angles are measured in radians, a circle's arc length and sector area follow directly: l = r*theta for arc length and Area = (1/2)*r^2*theta for the area of a sector, both derived from simple proportions between angles and their arcs.

The heart of the unit is the definition of the six trigonometric ratios -- sine, cosine, tangent, cotangent, secant, and cosecant -- using a unit circle, which extends these ratios to angles of ANY size, not just the acute angles of a right triangle. General (coterminal) angles and the idea of standard position let any angle be classified into one of four quadrants, whose signs of ratios follow the ASTC pattern. The unit shows how to recall exact values at 45, 30, 60 degrees and at the quadrantal angles 0, 90, 180, 270, 360 degrees, how to find every other ratio once a single ratio is known, and closes with the three fundamental (Pythagorean) trigonometric identities and their first real-world application: finding heights and distances using the angle of elevation and angle of depression.

Learning Objectives

- Measure an angle in degrees, minutes, and seconds, and convert between D M S form and decimal degrees
- Define a radian and prove the relationship between radians and degrees (180 degrees = pi radians)
- Establish and apply the rule l = r*theta for the length of a circular arc
- Prove and apply the formula Area of a sector = (1/2)*r^2*theta
- Define a general angle (coterminal angles) and locate an angle in standard position

- Recognize quadrants and quadrantal angles
- Define the six trigonometric ratios and their reciprocals using a unit circle
- Recall the exact values of trigonometric ratios for 45, 30, and 60 degrees
- Recognize the signs of trigonometric ratios in each quadrant, and find the remaining ratios when one is given
- Calculate trigonometric ratios at 0, 90, 180, 270, 360 degrees, prove and apply trigonometric identities, and solve angle of elevation/depression problems

Key Concepts

7.1 Measurement of an Angle: Degrees and Radians

An angle is the union of two non-collinear rays (its arms) sharing a common endpoint (its vertex); the original ray position is the initial side and the final position after rotation is the terminal side -- anticlockwise rotation gives a positive angle, clockwise gives a negative angle. In the sexagesimal system, the circumference of a circle is divided into 360 equal arcs, each subtending one degree (1 deg) at the centre; 60 seconds (60'') make one minute (1'), 60 minutes make one degree, and 90 degrees makes one right angle. An angle is in standard position when its vertex is at the origin and its initial side lies along the positive x-axis. Converting between D M S form and decimal degrees is done by dividing minutes by 60 and seconds by 3600 (to go to decimal), or by multiplying the decimal part by 60 repeatedly (to go back to D M S).

The circular (radian) system, used throughout higher mathematics, defines one radian as the angle subtended at the centre of a circle by an arc whose length equals the circle's radius. Since a circle's circumference is 2*pi*r, a complete revolution measures 2*pi*r/r = 2*pi radians, giving the key relationship 360 degrees = 2*pi radians, or equivalently 180 degrees = pi radians. This yields the conversion formulas: x degrees = x(pi/180) radians, and y radians = y(180/pi) degrees -- so 1 degree is about 0.0175 radians and 1 radian is about 57.296 degrees (57 deg 17' 45'').

7.2 Sector of a Circle: Arc Length and Area

A part of a circle's circumference is called an arc; a region bounded by an arc and a chord is a segment; a region bounded by two radii and an arc is a sector. Because the central angles subtended by arcs of a circle are proportional to the lengths of those arcs, comparing an arc of length l (subtending theta radians) to the arc of one full radian (length r) gives theta/1 = l/r, so l = r*theta -- the rule connecting arc length, radius, and central angle, valid only when theta is measured in radians.

The area of a sector follows a similar proportion: the ratio of a sector's area to the full circle's area (pi*r^2) equals the ratio of the sector's angle theta to the full angle of the circle (2*pi radians). Solving (Area of sector)/(pi*r^2) = theta/(2*pi) gives Area of sector = (theta/2*pi) * pi*r^2 = (1/2)*r^2*theta, again with theta in radians.

7.3 General Angles, Standard Position, and Quadrants

Two or more angles that share the same initial side and the same terminal side are called coterminal angles -- a terminal side returns to its original position after every complete revolution of 2*pi radians (360 degrees), so if theta is coterminal with some angle, then theta + 360*k degrees or theta + 2*k*pi radians, for any integer k, is also coterminal with it. This gives the idea of a general angle: theta = 2*k*pi + theta, k is an integer.

The x-axis and y-axis divide the coordinate plane into four quadrants, meeting at the origin O. An angle in standard position is said to lie in whichever quadrant its terminal side falls in: Quadrant I is 0 to 90 degrees, Quadrant II is 90 to 180 degrees, Quadrant III is 180 to 270 degrees, and Quadrant IV is 270 to 360 degrees. If the terminal side of an angle in standard position falls exactly on the x-axis or y-axis (i.e. the angle is 90, 180, 270, or 360 degrees), it is called a quadrantal angle.

7.4 Trigonometric Ratios and the Unit Circle

Let theta be the radian measure of an angle in standard position, and let P(x, y) be the point where its terminal side meets the unit circle (radius 1, centred at the origin). Sine and cosine are defined directly as the coordinates of P: cos theta = x and sin theta = y. The remaining four ratios follow: tan theta = y/x = sin theta/cos theta (x not 0), cot theta = x/y = cos theta/sin theta (y not 0), sec theta = 1/x = 1/cos theta (x not 0), and cosec theta = 1/y = 1/sin theta (y not 0) -- giving the reciprocal identities sin theta = 1/cosec theta, cos theta = 1/sec theta, and tan theta = 1/cot theta. For a point (x, y) not necessarily on the unit circle, r = sqrt(x^2+y^2) is used instead, so sin theta = y/r, cos theta = x/r, tan theta = y/x, and so on.

The exact values at 45, 30, and 60 degrees are found using two special right triangles. Using an isosceles right triangle with legs a = b = 1 (so hypotenuse c = sqrt(2)) gives sin 45 = cos 45 = 1/sqrt(2) and tan 45 = 1. Using an equilateral triangle with side 2, bisected into two 30-60-90 triangles with height sqrt(3), gives sin 30 = 1/2, cos 30 = sqrt(3)/2, tan 30 = 1/sqrt(3), and sin 60 = sqrt(3)/2, cos 60 = 1/2, tan 60 = sqrt(3) -- with the reciprocal ratios (cosec, sec, cot) following directly from these.

7.5 Signs of Trigonometric Ratios and Finding Remaining Ratios

Since r = sqrt(x^2+y^2) is always positive, the sign of each ratio depends only on the signs of x and y in the quadrant containing theta's terminal side. In Quadrant I, x and y are both positive, so all six ratios are positive. In Quadrant II, x is negative and y is positive, so only sin theta (and cosec theta) are positive. In Quadrant III, both x and y are negative, so only tan theta (and cot theta) are positive. In Quadrant IV, x is positive and y is negative, so only cos theta (and sec theta) are positive. This pattern is remembered with the mnemonic ASTC ('Add Sugar To Coffee'): All positive in QI, Sine positive in QII, Tangent positive in QIII, Cosine positive in QIV.

If one trigonometric ratio is known (along with which quadrant theta lies in, to fix the signs), every other ratio can be found. One method treats the known ratio as a fraction of two sides of a right triangle, finds the third side with the Pythagorean theorem, and reads off the remaining ratios directly. A second method uses the reciprocal identities together with the Pythagorean identities (covered next) to solve algebraically for each remaining ratio in terms of the one given. The same unit-circle approach gives exact values at the quadrantal angles: at theta = 0 the point is (1,0), at 90 it is (0,1), at 180 it is (-1,0), and at

270 it is (0,-1) -- so ratios with a zero denominator (such as tan 90 or sec 90) are undefined at these angles, while theta = 360 repeats the values of theta = 0.

7.6 Trigonometric Identities and Angle of Elevation/Depression

For any point P(x,y) on the terminal side of theta with $r = sqrt(x^2+y^2)$, the Pythagorean theorem gives $x^2 + y^2 = r^2$. Dividing this equation by r^2 gives $cos^2(theta) + sin^2(theta) = 1$ (the first Pythagorean identity). Dividing instead by x^2 gives $1 + tan^2(theta) = sec^2(theta)$, and dividing by y^2 gives $1 + cot^2(theta) = cosec^2(theta)$ -- together these three are called the Pythagorean identities, and they are the main tool for simplifying trigonometric expressions and proving that a left-hand-side expression equals a right-hand-side expression (usually by first rewriting everything in terms of sin theta and cos theta).

One of trigonometry's classic applications is finding heights and distances without directly measuring them. If a horizontal line is drawn through an observer's eye, the angle of elevation is the angle between that horizontal line and the line of sight up to an object above it; the angle of depression is the angle between the horizontal line and the line of sight down to an object below it. Both are measured FROM the horizontal line. Once such an angle is known along with one distance (e.g. a shadow length), the tangent ratio (tan angle = opposite/adjacent) is used to solve for the unknown height or distance.

Important Definitions

What is an angle?

The union of two non-collinear rays (arms) sharing a common endpoint (vertex); rotating one ray to another anticlockwise gives a positive angle, clockwise gives a negative angle.

What is one degree?

The angle subtended at the centre of a circle by 1/360 of its circumference; 60 minutes make one degree and 60 seconds make one minute.

What is one radian?

The angle subtended at the centre of a circle by an arc whose length is equal to the circle's radius.

What is the relationship between degrees and radians?

180 degrees = pi radians, giving x degrees = x(pi/180) radians and y radians = y(180/pi) degrees.

What is a sector of a circle?

The region of a circle bounded by two radii and the arc between them.

What are coterminal angles?

Two or more angles that share the same initial side and the same terminal side, differing by a whole number of complete revolutions (360k degrees or 2k*pi radians, k an integer).

What is an angle in standard position?

An angle whose vertex is at the origin and whose initial side lies along the positive x-axis of a coordinate system.

What is a quadrantal angle?

An angle in standard position whose terminal side falls exactly on the x-axis or y-axis -- i.e. 90, 180, 270, or 360 degrees.

How are sine and cosine defined using the unit circle?

For angle theta in standard position with terminal side meeting the unit circle at P(x,y): cos theta = x (the x-coordinate of P) and sin theta = y (the y-coordinate of P).

What is the angle of elevation?

The angle between a horizontal line through an observer and the line of sight up to an object located above that horizontal line.

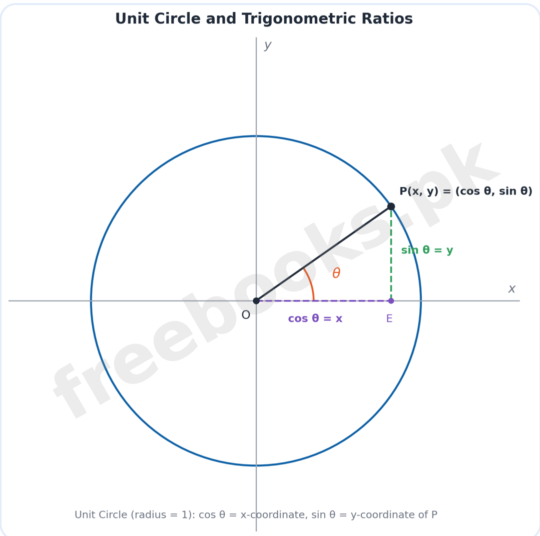
Key Facts and Relations

Topic	Key Fact / Relation
Degree to radian	x degrees = x(pi/180) radians
Radian to degree	y radians = y(180/pi) degrees
Arc length	l = r*theta (theta in radians)
Area of a sector	Area = (1/2) r^2 theta (theta in radians)

Topic	Key Fact / Relation
Unit circle definitions	cos theta = x, sin theta = y
Quotient identities	tan theta = sin theta / cos theta, cot theta = cos theta / sin theta
Reciprocal identities	sec theta = 1/cos theta, cosec theta = 1/sin theta, cot theta = 1/tan theta
Pythagorean identity I	cos^2(theta) + sin^2(theta) = 1
Pythagorean identity II	1 + tan^2(theta) = sec^2(theta)
Pythagorean identity III	1 + cot^2(theta) = cosec^2(theta)
Coterminal (general) angle	theta + 360k degrees or theta + 2k*pi radians, k an integer
Special values (30, 45, 60 degrees)	sin30=1/2, sin45=1/sqrt(2), sin60=sqrt(3)/2 (cos mirrors: cos30=sqrt(3)/2, cos60=1/2)

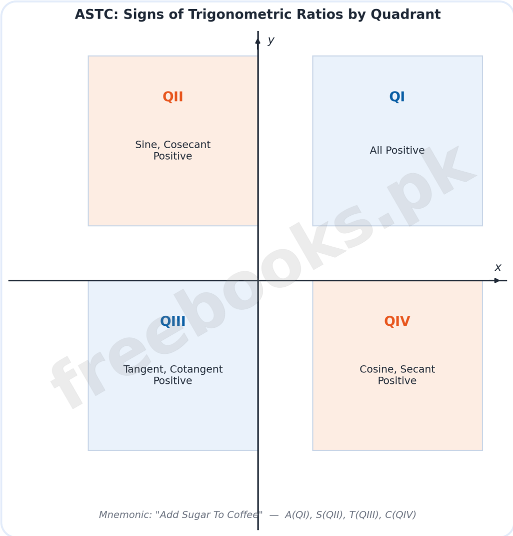
Diagrams

Unit Circle and Trigonometric Ratios: A unit circle with an angle theta in standard position, showing point P(x,y) on the circle, the right triangle formed by dropping a perpendicular to the x-axis, and how cos theta and sin theta correspond to the x- and y-coordinates of P



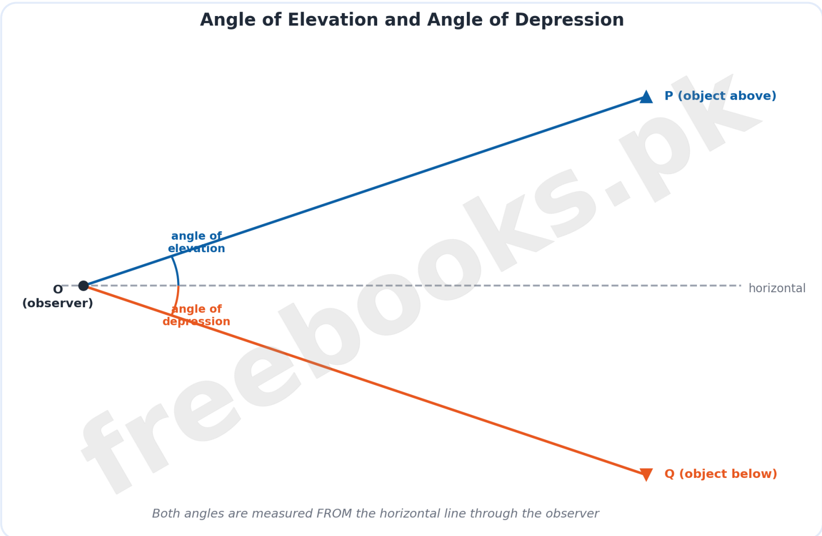
Unit Circle and Trigonometric Ratios

ASTC: Signs of Trigonometric Ratios by Quadrant: A quadrant diagram showing which trigonometric ratios are positive in each of the four quadrants, following the ASTC (All, Sine, Tangent, Cosine) rule



ASTC: Signs of Trigonometric Ratios by Quadrant

Angle of Elevation and Angle of Depression: A diagram showing an observer's horizontal line of sight, with the angle of elevation measured up to an object above and the angle of depression measured down to an object below



Angle of Elevation and Angle of Depression

Short Questions & Answers

Define one radian.

One radian is the angle subtended at the centre of a circle by an arc whose length is equal to the circle's radius.

What is the relationship between 180 degrees and pi radians?

180 degrees = pi radians; this is the key relationship used to convert between degrees and radians.

Define coterminal angles.

Coterminal angles are two or more angles that share the same initial side and the same terminal side, differing by a whole number of complete revolutions.

What is a quadrantal angle?

A quadrantal angle is an angle in standard position whose terminal side falls exactly on the x-axis or y-axis, i.e. 90, 180, 270, or 360 degrees.

State the ASTC rule.

ASTC ('Add Sugar To Coffee') states that All ratios are positive in Quadrant I, Sine (and cosecant) in Quadrant II, Tangent (and cotangent) in Quadrant III, and Cosine (and secant) in Quadrant IV.

Define angle of elevation.

The angle of elevation is the angle between a horizontal line through an observer and the line of sight up to an object located above that line.

Long Questions & Answers

Explain how an angle is measured in the sexagesimal and circular systems, and how the length of an arc and the area of a sector are found.

What is the sexagesimal system of angle measurement?

The sexagesimal system divides a circle's circumference into 360 equal arcs, each subtending one degree; 60 seconds make one minute and 60 minutes make one degree, with 90 degrees equal to one right angle and 360 degrees equal to one complete revolution.

How is an angle converted between D M S form and decimal degrees?

To convert to decimal, divide the minutes by 60 and the seconds by 3600 and add these to the whole degrees (e.g. 25 deg 30' = 25 + 30/60 = 25.5 degrees). To convert back, multiply the decimal fraction of a degree by 60 to get minutes, and any remaining decimal fraction of a minute by 60 to get seconds.

What is a radian, and how are degrees and radians related?

A radian is the angle subtended by an arc equal in length to the circle's radius. Since a full revolution equals 2*pi radians and also 360 degrees, the key relationship is 180 degrees = pi radians, giving x degrees = x(pi/180) radians and y radians = y(180/pi) degrees.

How is the length of a circular arc calculated?

Because central angles are proportional to their arc lengths, comparing an arc l (subtending theta radians) to the one-radian arc (length r) gives l = r*theta, where theta must be in radians.

How is the area of a sector of a circle calculated?

By proportion, (area of sector)/(area of circle) = theta/(2*pi), so Area of sector = (theta/2*pi) * pi*r^2 = (1/2)*r^2*theta, again with theta measured in radians.

Explain how the six trigonometric ratios are defined using the unit circle, and how their signs and values are determined in different quadrants.

How are the six trigonometric ratios defined using a point on the unit circle?

For angle theta in standard position with terminal side meeting the unit circle at P(x,y): cos theta = x and sin theta = y directly; then tan theta = y/x, cot theta = x/y, sec theta = 1/x, and cosec theta = 1/y, giving the reciprocal identities connecting each ratio to its reciprocal.

What are the exact values of the trigonometric ratios for 30, 45, and 60 degrees?

From an isosceles right triangle (legs 1, hypotenuse sqrt(2)): sin45 = cos45 = 1/sqrt(2), tan45 = 1. From an equilateral triangle of side 2 bisected into 30-60-90 triangles (height sqrt(3)): sin30 = 1/2, cos30 = sqrt(3)/2, tan30 = 1/sqrt(3), and sin60 = sqrt(3)/2, cos60 = 1/2, tan60 = sqrt(3).

How is the sign of a trigonometric ratio determined in each quadrant (the ASTC rule)?

Since r is always positive, signs depend only on x and y in that quadrant: All ratios positive in QI (x,y both positive), Sine/cosecant positive in QII (x negative, y positive), Tangent/cotangent positive in QIII (x,y both negative), Cosine/secant positive in QIV (x positive, y negative) -- remembered as ASTC, 'Add Sugar To Coffee'.

How can the remaining trigonometric ratios be found if one ratio is known?

Treat the known ratio as two sides of a right triangle, use the Pythagorean theorem to find the third side, then read off the remaining ratios directly -- using the given quadrant (or sign information) to assign the correct positive or negative sign to each result.

What are the Pythagorean identities, and how are they used to simplify trigonometric expressions?

From x^2+y^2=r^2, dividing by r^2 gives cos^2(theta)+sin^2(theta)=1; dividing by x^2 gives 1+tan^2(theta)=sec^2(theta); dividing by y^2 gives 1+cot^2(theta)=cosec^2(theta). These three identities are used to rewrite trigonometric expressions in simpler or equivalent forms, usually by first converting everything to sin theta and cos theta.

Multiple Choice Questions (MCQs)

One radian is the angle subtended by an arc whose length equals:

A. The diameter

B. The radius

C. The circumference

D. Half the radius

Answer: B. The radius

Explanation: One radian is defined as the angle subtended at the centre of a circle by an arc equal in length to the radius.

180 degrees is equal to:

A. pi/2 radians

B. pi radians

C. 2*pi radians

D. pi/4 radians

Answer: B. pi radians

Explanation: The key relationship linking the two angle systems is 180 degrees = pi radians.

The length of a circular arc is given by the formula:

A. l = r*theta

B. l = r/theta

C. l = r^2*theta

D. l = 2*pi*r

Answer: A. l = r*theta

Explanation: Arc length is given by l = r*theta, where theta is the central angle measured in radians.

The area of a sector of a circle is given by:

A. pi*r^2

B. (1/2)*r^2*theta

C. r*theta

D. 2*pi*r*theta

Answer: B. (1/2)*r^2*theta

Explanation: The area of a sector is Area = (1/2)*r^2*theta, with theta measured in radians.

Two angles with the same initial and terminal sides are called:

A. Quadrantal angles

B. Coterminal angles

C. Standard angles

D. Reference angles

Answer: B. Coterminal angles

Explanation: Angles sharing the same initial and terminal sides are called coterminal angles.

On the unit circle, cos theta is defined as:

A. The y-coordinate of P

B. The x-coordinate of P

C. The radius

D. The arc length

Answer: B. The x-coordinate of P

Explanation: By definition on the unit circle, cos theta is the x-coordinate of the point P where the terminal side meets the circle.

In which quadrant are all six trigonometric ratios positive?

A. I

B. II

C. III

D. IV

Answer: A. I

Explanation: In Quadrant I, both x and y are positive, so all six trigonometric ratios are positive there.

The value of tan 45 degrees is:

A. 0

B. 1

C. sqrt(3)

D. Undefined

Answer: B. 1

Explanation: From the isosceles right triangle with legs 1 and 1, tan45 = 1/1 = 1.

Which of the following is a Pythagorean identity?

A. sin theta = 1/cosec theta

B. cos^2(theta) + sin^2(theta) = 1

C. tan theta = sin theta/cos theta

D. l = r*theta

Answer: B. cos^2(theta) + sin^2(theta) = 1

Explanation: cos^2(theta) + sin^2(theta) = 1 is one of the three Pythagorean identities, derived from x^2+y^2=r^2.

The angle measured downward from a horizontal line to an object is called the angle of:

A. Elevation

B. Standard position

C. Depression

D. Rotation

Answer: C. Depression

Explanation: The angle measured from the horizontal down to an object below is called the angle of depression.

Quick Revision Summary

- Angle: union of two rays sharing a vertex; positive if anticlockwise, negative if clockwise
- Degree: 60''=1', 60'=1 degree, 90 degrees=1 right angle, 360 degrees=1 revolution
- Radian: angle subtended by an arc of length = radius | 180 degrees = pi radians
- Conversion: x degrees = x(pi/180) radians | y radians = y(180/pi) degrees
- Arc length l = r*theta | Area of sector = (1/2)*r^2*theta (theta in radians)
- Coterminal (general) angle: theta + 360k degrees or theta + 2k*pi radians, k an integer
- Standard position: vertex at origin, initial side on positive x-axis
- Quadrants: I(0-90deg), II(90-180deg), III(180-270deg), IV(270-360deg) | Quadrantal angles: 90,180,270,360 deg
- Unit circle: cos theta = x, sin theta = y | tan theta = y/x, cot theta = x/y, sec theta = 1/x, cosec theta = 1/y
- ASTC rule: All +ve in QI, Sine(&cosec) +ve in QII, Tangent(&cot) +ve in QIII, Cosine(&sec) +ve in QIV
- Pythagorean identities: cos^2(theta)+sin^2(theta)=1, 1+tan^2(theta)=sec^2(theta), 1+cot^2(theta)=cosec^2(theta)
- Angle of elevation (looking up) and angle of depression (looking down), both measured from a horizontal line

Exam Tips

- Always confirm whether an angle is given in degrees or radians before applying a formula -- mixing the two is the most common trigonometry mistake
- Memorize just the two special triangles (45-45-90 and 30-60-90) instead of a full values table -- every special-angle value can be derived from them
- Use the mnemonic 'Add Sugar To Coffee' (ASTC) to instantly recall which two ratios are positive in each quadrant
- When one ratio is given, sketch a quick right triangle using its numerator and denominator as two sides, then find the third side with the Pythagorean theorem
- For identity proofs, convert everything to sin theta and cos theta first -- most identities simplify almost immediately once written this way
- In elevation/depression problems, draw the horizontal reference line first -- the angle is always measured FROM that line, never from the vertical

Unit 8: Projection of a Side of a Triangle

This short but important unit formalizes the idea of the projection of one side of a triangle onto another, and uses it to extend the Pythagorean theorem to triangles that are NOT right-angled. Three theorems are proved: the obtuse-angle theorem (for the side opposite an obtuse angle), the acute-angle theorem (for the side opposite an acute angle), and Apollonius' theorem (relating the two sides of a triangle to the median that bisects the third side). Each is proved by drawing a perpendicular (altitude) and applying the ordinary Pythagorean theorem twice -- once in each of the two right triangles the altitude creates.

These theorems matter because the plain Pythagorean theorem only applies to right triangles; the projection theorems generalize it so that unknown sides, angles, and medians can be found in ANY triangle once one angle is known to be acute or obtuse. This is the same idea, expressed geometrically, that the Law of Cosines expresses algebraically in later mathematics -- a bridge between the projection concept here and the trigonometric ratios covered in the previous unit.

Learning Objectives

- Define the projection of a point, and of a line segment, on another line segment
- State and prove Theorem 1: the obtuse-angle projection theorem
- State and prove Theorem 2: the acute-angle projection theorem
- State and prove Theorem 3 (Apollonius' theorem): the relationship between two sides and the median to the third side
- Apply the projection theorems to compute an unknown side, angle, or median of a triangle
- Recognize how the sign of the projection term differs between the obtuse-angle and acute-angle theorems
- Solve problems that combine the projection theorems with basic properties of isosceles triangles

Key Concepts

8.1 The Projection of a Line Segment

The projection of a point C on a line segment AB is the foot of the perpendicular drawn from C to AB -- if CD is perpendicular to AB, then D (the point where the perpendicular meets AB) is the projection of C on AB. Extending this, the projection of a line segment CD on a line segment AB is the portion EF of AB intercepted between the feet of the perpendiculars dropped from C and from D onto AB.

A special case worth noting: if the segment CD being projected is itself perpendicular to AB, its projection is not a line segment at all but a single point on AB -- described as a projection of 'zero dimension'. This projection idea is the geometric foundation for both projection theorems that follow, where the projection of one side of a triangle onto another (or onto its extension) becomes the extra term that generalizes the Pythagorean theorem.

8.2 Theorem 1: The Obtuse-Angle Theorem

Theorem 1 states: in an obtuse-angled triangle, the square on the side opposite the obtuse angle equals the sum of the squares on the sides containing the obtuse angle, PLUS twice the rectangle contained by one of those sides and the projection of the other side upon it. Symbolically, if triangle ABC has an obtuse angle at A, with BC=a, CA=b, AB=c, and CD is drawn perpendicular to BA produced (meeting it at D, so that AD=x is the projection of AC on BA produced), then $a^2 = b^2 + c^2 + 2cx$.

The proof applies the Pythagorean theorem twice: in right triangle CDA, $b^2 = x^2 + h^2$ (where h = CD); in right triangle CDB, since BD = BA + AD = c + x, $a^2 = (c+x)^2 + h^2 = c^2 + 2cx + x^2 + h^2$. Substituting $b^2 = x^2 + h^2$ from the first equation into the second gives $a^2 = c^2 + 2cx + b^2$, i.e. $a^2 = b^2 + c^2 + 2cx$ -- the extra '+2cx' term is exactly what distinguishes this from the ordinary Pythagorean theorem, and it arises because D falls OUTSIDE segment AB when the angle at A is obtuse.

8.3 Theorem 2: The Acute-Angle Theorem

Theorem 2 states: in any triangle, the square on the side opposite an acute angle equals the sum of the squares on the sides containing that acute angle, MINUS twice the rectangle contained by one of those sides and the projection of the other side upon it. With the same labelling (BC=a, CA=b, AB=c) but now with CD perpendicular to AB itself (D falling INSIDE segment AB, so AD=x), the result is $a^2 = b^2 + c^2 - 2cx$ -- identical to Theorem 1 except for the sign of the projection term.

The proof mirrors Theorem 1's: in right triangle CDA, $b^2 = x^2 + h^2$; in right triangle CDB, since $BD = c - x$ (because D lies between B and A), $a^2 = (c-x)^2 + h^2 = c^2 - 2cx + x^2 + h^2$. Substituting gives $a^2 = b^2 + c^2 - 2cx$. A useful corollary appears when the triangle is isosceles: if AB = AC and BE is drawn perpendicular to AC (with CE the projection of BC on AC), Theorem 2 simplifies neatly to $(BC)^2 = 2\cdot(AC)\cdot(CE)$, since the $(AB)^2$ and $(AC)^2$ terms cancel.

8.4 Theorem 3: Apollonius' Theorem (the Median Theorem)

Apollonius' theorem states: in any triangle, the sum of the squares on any two sides equals twice the square on half the third side, together with twice the square on the median which bisects that third side. If AD is the median from A to side BC (so BD = DC = half of BC), the theorem gives $(AB)^2 + (AC)^2 = 2(BD)^2 + 2(AD)^2$.

The proof is a direct combination of Theorems 1 and 2: draw AF perpendicular to BC. In triangle ADB, angle ADB is acute at D, so by Theorem 2, $(AB)^2 = (BD)^2 + (AD)^2 - 2\cdot BD\cdot FD$. In triangle ADC, angle ADC is obtuse at D, so by Theorem 1, $(AC)^2 = (CD)^2 + (AD)^2 + 2\cdot CD\cdot FD$; since CD = BD, this becomes $(AC)^2 = (BD)^2 + (AD)^2 + 2\cdot BD\cdot FD$. Adding the two equations, the '$-2\cdot BD\cdot FD$' and '$+2\cdot BD\cdot FD$' terms cancel exactly, leaving $(AB)^2 + (AC)^2 = 2(BD)^2 + 2(AD)^2$ -- letting the length of a median be calculated directly from the three side lengths of a triangle, without needing to know any of its angles.

Important Definitions

What is the projection of a point on a line segment?

The foot of the perpendicular drawn from that point to the line segment.

What is the projection of a line segment on another line segment?
The portion of the second segment intercepted between the feet of the perpendiculars dropped from the two endpoints of the first segment.

What does Theorem 1 (the obtuse-angle theorem) state?
In an obtuse-angled triangle, the square on the side opposite the obtuse angle equals the sum of the squares on the sides containing it, plus twice the rectangle of one side and the projection of the other upon it: $a^2 = b^2 + c^2 + 2cx$.

What does Theorem 2 (the acute-angle theorem) state?
In any triangle, the square on the side opposite an acute angle equals the sum of the squares on the sides containing it, minus twice the rectangle of one side and the projection of the other upon it: $a^2 = b^2 + c^2 - 2cx$.

What is Apollonius' theorem?
In any triangle, the sum of the squares on any two sides equals twice the square on half the third side, plus twice the square on the median that bisects it: $(AB)^2 + (AC)^2 = 2(BD)^2 + 2(AD)^2$.

What is a median of a triangle?
A line segment joining a vertex of a triangle to the midpoint of the opposite side.

Why does Theorem 1 have a plus sign before the projection term?
Because the obtuse angle at A pushes the foot of the perpendicular D outside segment AB, so BD = BA + AD, which produces a '+2cx' term when squared.

Why does Theorem 2 have a minus sign before the projection term?
Because the acute angle at A keeps the foot of the perpendicular D inside segment AB, so BD = BA − AD, which produces a '−2cx' term when squared.

What is the projection of a vertical (perpendicular) line segment on a base line?
A single point of zero dimension, since both of its endpoints project to the same foot of the perpendicular.

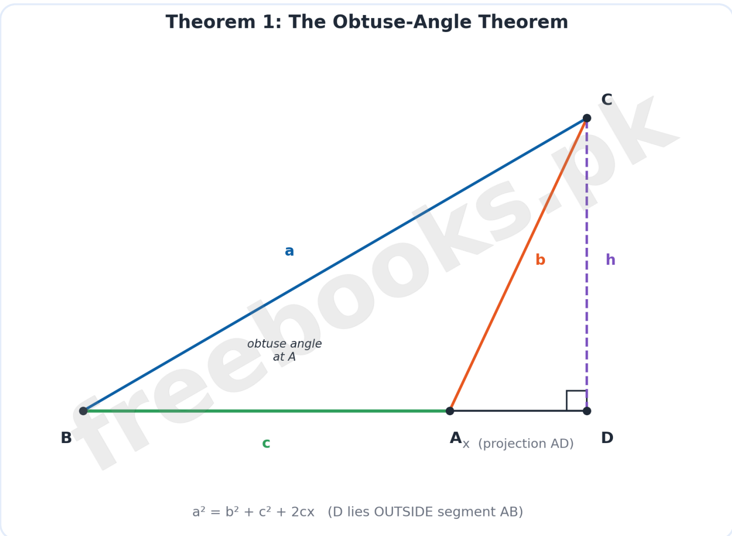
What theorem is normally used to find the length of a triangle's median from its three side lengths?
Apollonius' theorem (Theorem 3), which relates the median directly to the three sides without requiring any angle.

Key Facts and Relations

Topic	Key Fact / Relation
Theorem 1 (obtuse-angle theorem)	$a^2 = b^2 + c^2 + 2cx$ (x = projection of AC on BA produced)
Theorem 2 (acute-angle theorem)	$a^2 = b^2 + c^2 - 2cx$ (x = projection of AC on AB)
Apollonius' theorem (Theorem 3)	$(AB)^2 + (AC)^2 = 2(BD)^2 + 2(AD)^2$ (AD = median, BD = DC = half of BC)
Isosceles corollary of Theorem 1	If AB = AC (obtuse at A): $(BC)^2 = 2\cdot(AB)\cdot(BD)$
Isosceles corollary of Theorem 2	If AB = AC (acute at C): $(BC)^2 = 2\cdot(AC)\cdot(CE)$
Pythagorean theorem (used inside every proof)	$(\text{hypotenuse})^2 = (\text{leg } 1)^2 + (\text{leg } 2)^2$
Projection and cosine (bridge to trigonometry)	projection $AD = AC \cdot \cos(\angle A)$

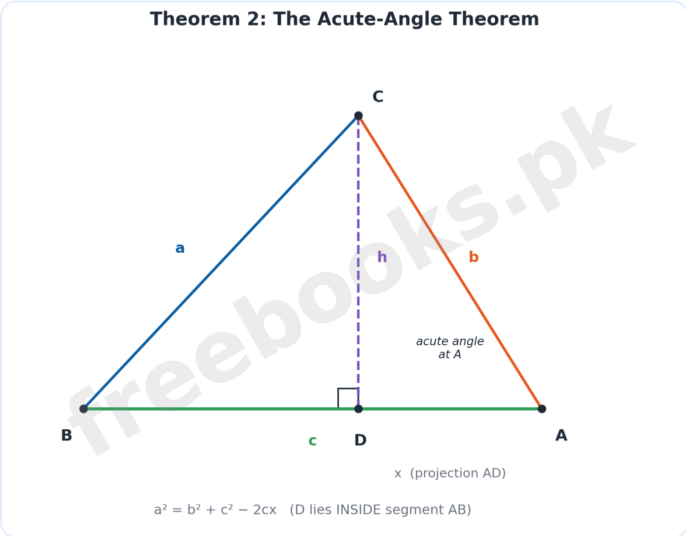
Diagrams

Theorem 1: The Obtuse-Angle Theorem: A triangle ABC with an obtuse angle at A, showing CD drawn perpendicular to BA produced, with the projection AD = x labelled alongside sides a, b, c and height h



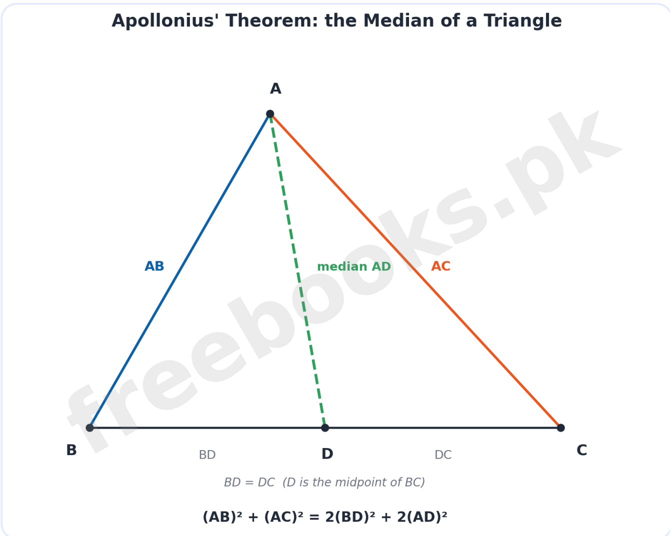
Theorem 1: The Obtuse-Angle Theorem

Theorem 2: The Acute-Angle Theorem: A triangle ABC with an acute angle at A, showing CD drawn perpendicular to AB, with the projection AD = x labelled alongside sides a, b, c and height h



Theorem 2: The Acute-Angle Theorem

Apollonius' Theorem: the Median of a Triangle: A triangle ABC with median AD drawn to the midpoint D of side BC, showing BD = DC and illustrating the relationship proved by Apollonius' theorem



Apollonius' Theorem: the Median of a Triangle

Short Questions & Answers

Define the projection of a point on a line segment.

The projection of a point on a line segment is the foot of the perpendicular drawn from that point to the segment.

State Theorem 1 (the obtuse-angle theorem).

In an obtuse-angled triangle, the square on the side opposite the obtuse angle equals the sum of the squares on the sides containing it, plus twice the rectangle of one side and the projection of the other upon it.

State Theorem 2 (the acute-angle theorem).

In any triangle, the square on the side opposite an acute angle equals the sum of the squares on the sides containing it, minus twice the rectangle of one side and the projection of the other upon it.

State Apollonius' theorem.

In any triangle, the sum of the squares on any two sides equals twice the square on half the third side, plus twice the square on the median which bisects it.

What is a median of a triangle?

A median is a line segment joining a vertex of a triangle to the midpoint of the side opposite that vertex.

In Theorem 2's setup, what does x represent?

x represents AD, the projection of side AC onto side AB, found by dropping a perpendicular from C to AB.

Long Questions & Answers

State and explain Theorem 1 (the obtuse-angle projection theorem) and Theorem 2 (the acute-angle projection theorem), including how their proofs use the Pythagorean theorem.

What does the obtuse-angle theorem (Theorem 1) state?

In an obtuse-angled triangle, the square on the side opposite the obtuse angle equals the sum of the squares on the sides containing that angle, plus twice the rectangle contained by one of those sides and the projection of the other side upon it: $a^2 = b^2 + c^2 + 2cx$.

How is Theorem 1 proved using the Pythagorean theorem?

Drawing CD perpendicular to BA produced gives two right triangles: CDA gives $b^2 = x^2 + h^2$, and CDB gives $a^2 = (c+x)^2 + h^2$ since BD = BA + AD. Substituting the first equation into the second gives $a^2 = b^2 + c^2 + 2cx$.

What does the acute-angle theorem (Theorem 2) state?

In any triangle, the square on the side opposite an acute angle equals the sum of the squares on the sides containing that angle, minus twice the rectangle contained by one of those sides and the projection of the other side upon it: $a^2 = b^2 + c^2 - 2cx$.

How is Theorem 2 proved, and why does the sign differ from Theorem 1?

The same two right triangles are used, but now D falls INSIDE segment AB (since the angle at A is acute), so $BD = c - x$ instead of $c + x$. Squaring this produces a '$-2cx$' term instead of '$+2cx$', which is the only difference between the two theorems.

How are these two theorems applied to solve a triangle when one angle and two sides are known?

First determine whether the given angle is acute or obtuse, then apply the matching theorem (Theorem 2 or Theorem 1) using the known sides and the projection (often found via cosine or a given length), solving directly for the unknown third side.

Explain Apollonius' theorem (Theorem 3), including how it is proved from Theorems 1 and 2, and what it is used for.

What does Apollonius' theorem state?

In any triangle, the sum of the squares on any two sides is equal to twice the square on half the third side together with twice the square on the median which bisects that third side: $(AB)^2 + (AC)^2 = 2(BD)^2 + 2(AD)^2$.

What is a median, and why does the theorem require it to bisect the third side?

A median joins a vertex to the midpoint of the opposite side. The theorem specifically needs BD = DC (D is the midpoint of BC) because the proof relies on both sub-triangles sharing the same base length BD = DC.

How does the proof of Apollonius' theorem combine Theorem 1 and Theorem 2?

The median AD splits the triangle into ADB (where angle ADB is acute, so Theorem 2 applies) and ADC (where angle ADC is obtuse, so Theorem 1 applies). Writing both equations and adding them cancels the opposite-signed projection terms, leaving only the squares.

Why does the construction draw a perpendicular from the vertex to the third side?

The perpendicular AF (from A to BC) is what allows Theorems 1 and 2 to be applied to the two sub-triangles ADB and ADC in the first place, since both theorems require a perpendicular dropped to identify a projection.

What kind of problems does Apollonius' theorem let us solve?

It allows the length of a median to be calculated directly from the three side lengths of a triangle (or vice versa, one side to be found given the other two and the median), without needing to know any angle of the triangle.

Multiple Choice Questions (MCQs)

The projection of a point C on a line segment AB is:

A. The foot of the perpendicular from C to AB
B. The midpoint of AB
C. Point A itself
D. The length of AB
Answer: A. The foot of the perpendicular from C to AB
Explanation: The projection of a point on a line is defined as the foot of the perpendicular dropped from that point onto the line.

In an obtuse triangle with the obtuse angle at A, Theorem 1 states $(BC)^2$ equals:

A. $(AC)^2+(AB)^2-2(AB)(AD)$

B. $(AC)^2+(AB)^2+2(AB)(AD)$

C. $(AC)^2-(AB)^2$

D. $2(AB)^2$

Answer: B. $(AC)^2+(AB)^2+2(AB)(AD)$

Explanation: Theorem 1 (the obtuse-angle theorem) adds twice the rectangle of one side and the projection: $a^2 = b^2 + c^2 + 2cx$.

In Theorem 2 (the acute-angle case), the sign before the projection term is:

A. Plus

B. Minus

C. Multiplication

D. Division

Answer: B. Minus

Explanation: Theorem 2 subtracts the projection term: $a^2 = b^2 + c^2 - 2cx$, since the foot of the perpendicular falls inside the side.

Apollonius' theorem relates the sum of squares of two sides of a triangle to:

A. The perimeter

B. The area

C. Twice the square on half the third side plus twice the square on the median

D. The product of all three sides

Answer: C. Twice the square on half the third side plus twice the square on the median

Explanation: Apollonius' theorem states $(AB)^2+(AC)^2 = 2(BD)^2+2(AD)^2$, connecting two sides to half the third side and the median.

A median of a triangle joins a vertex to:

A. The opposite vertex

B. The midpoint of the opposite side

C. The centroid only

D. The foot of the altitude

Answer: B. The midpoint of the opposite side

Explanation: By definition, a median joins a vertex of a triangle to the midpoint of the side opposite that vertex.

Theorem 1 and Theorem 2 are both proved using:

A. The Pythagorean theorem

B. Apollonius' theorem

C. The ASTC rule

D. The sine rule

Answer: A. The Pythagorean theorem

Explanation: Both projection theorems are proved by applying the Pythagorean theorem to the two right triangles formed by the altitude.

If AB = AC in an obtuse triangle with the obtuse angle at A, Theorem 1 simplifies to:

A. $(BC)^2=2(AB)(BD)$

B. $(BC)^2=(AB)^2$

C. $(BC)^2=0$

D. $(BC)^2=4(AB)^2$

Answer: A. $(BC)^2=2(AB)(BD)$

Explanation: When AB = AC, the two equal-side terms combine and Theorem 1 reduces to $(BC)^2 = 2(AB)(BD)$.

The projection of a vertical line segment onto a perpendicular base line is:

A. A line

B. A circle

C. A point of zero dimension

D. Undefined

Answer: C. A point of zero dimension

Explanation: When the segment being projected is itself perpendicular to the base, both its endpoints project to the same single point.

In proving Apollonius' theorem, the perpendicular AF is drawn from vertex A to:

A. The median

B. Side BC

C. Side AB

D. Side AC

Answer: B. Side BC

Explanation: The construction draws AF perpendicular to BC, allowing Theorems 1 and 2 to be applied to the two sub-triangles formed by the median.

Which theorem would you use to find the length of a triangle's median given all three sides?

A. Theorem 1

B. Theorem 2

C. Apollonius' theorem (Theorem 3)

D. The ASTC rule

Answer: C. Apollonius' theorem (Theorem 3)

Explanation: Apollonius' theorem directly relates the three sides of a triangle to the length of a median, making it the right tool for this problem.

Quick Revision Summary

- Projection of a point on a line = the foot of the perpendicular from that point to the line
- Projection of a segment CD on segment AB = the portion between the feet of the perpendiculars from C and D
- Theorem 1 (obtuse angle): $a^2 = b^2 + c^2 + 2cx$, where $x = AD$ is the projection of AC on BA produced
- Theorem 2 (acute angle): $a^2 = b^2 + c^2 - 2cx$, where $x = AD$ is the projection of AC on AB
- Both theorems are proved using the Pythagorean theorem applied to two right triangles sharing the altitude CD
- Apollonius' theorem: $(AB)^2 + (AC)^2 = 2(BD)^2 + 2(AD)^2$, where AD is the median and BD = DC
- Apollonius' theorem is proved by adding the Theorem 2 result (acute at D) and Theorem 1 result (obtuse at D) for the two sub-triangles formed by the median
- A median joins a vertex to the midpoint of the opposite side
- These theorems generalize the Pythagorean theorem to triangles that are not right-angled
- Isosceles special cases: $(BC)^2=2\cdot AB\cdot BD$ (obtuse at A) and $(BC)^2=2\cdot AC\cdot CE$ (acute case) when AB=AC

Exam Tips

- Always identify first whether the given angle is acute or obtuse -- that decides which theorem (and which sign) to use
- Draw the perpendicular (altitude) clearly and label the projection segment x before writing any equation
- Remember Theorem 1's '+2cx' and Theorem 2's '−2cx' are the ONLY difference between the two theorems -- everything else is identical
- For Apollonius' theorem problems, always check the given segment truly is a median (bisects the opposite side), not just any line
- When AB = AC (isosceles), both theorems simplify neatly -- look for this shortcut before grinding through the full formula
- Connect 'projection' language to cosine language (projection = adjacent side x cos of the angle) -- it ties this unit back to trigonometry

Unit 9: Chords of a Circle

This unit opens the geometry of circles by first fixing the basic vocabulary -- centre, radius, circumference, arc, chord, diameter, segment, sector, and central angle -- and then proving five theorems that describe exactly how a chord relates to the centre of its circle. Together these theorems explain why a triangle's circumcircle is unique, how a perpendicular from the centre always bisects a chord (and vice versa), and how the DISTANCE of a chord from the centre determines whether two chords are equal in length.

These results are the geometric toolkit used throughout the rest of the geometry units (tangents, arcs, and inscribed angles all build on chord properties). Each theorem is proved using triangle congruence (SSS, SAS, or the right-triangle hypotenuse-side postulate) applied to the triangles formed by joining the centre of the circle to the endpoints of a chord -- the same proof pattern repeats with small variations across all five theorems.

Learning Objectives

- Define a circle and identify its basic parts: centre, radius, circumference, arc, chord, diameter, segment, sector, and central angle
- Distinguish between an arc and a chord, and between a sector and a segment of a circle
- Prove that one and only one circle can pass through three non-collinear points
- Prove that a line drawn from the centre of a circle to bisect a chord (not a diameter) is perpendicular to that chord
- Prove that a perpendicular from the centre of a circle to a chord bisects the chord
- Prove that congruent chords of a circle are equidistant from the centre
- Prove that chords of a circle which are equidistant from the centre are congruent
- Apply these theorems to find unknown chord lengths, distances from the centre, or the diameter of a circle

Key Concepts

9.1 Basic Concepts of a Circle

A circle is the locus of a moving point P in a plane that stays always equidistant from a fixed point O; O is called the centre, the constant distance OP is the radius, and the path traced by P is the circumference, equal to 2*pi*r. Every radius of a given circle is equal in length, though a circle has only one centre. An arc is any portion of the circumference; a chord is a line segment joining any two points on the circumference, and a diameter is a chord that passes through the centre (so it evidently bisects the circle into two equal halves).

A segment is the region of a circle bounded by an arc and its corresponding chord -- any chord divides the circle into a larger major segment and a smaller minor segment. A sector, by contrast, is the region bounded by two radii and the arc between them -- any pair of radii divides the circle into a major sector and a minor sector. The central angle of a circle is the angle whose vertex is the centre and whose arms are the two radii drawn to the endpoints of an arc.

9.2 Theorem 1: One Circle Through Three Non-Collinear Points

Theorem 1 states: one and only one circle can pass through three non-collinear points. Given three non-collinear points A, B, and C, drawing the perpendicular bisectors of AB and of BC gives two lines that are not parallel (since A, B, C are non-collinear) and so must intersect at a unique point O. Every point on the perpendicular bisector of AB is equidistant from A and B, so OA = OB; similarly every point on the perpendicular bisector of BC is equidistant from B and C, so OB = OC. Combining these, OA = OB = OC, so a circle centred at O with radius OA passes through all three points -- and since O is the ONLY point common to both perpendicular bisectors, this circle is unique.

A direct application: since the diagonals of a rectangle are equal in length and bisect each other, the point O where they meet is equidistant from all four vertices, so exactly one circle (the rectangle's circumcircle) passes through all four vertices of any rectangle -- a special case of the same underlying idea, since any three of the four vertices already determine that unique circle.

9.3 Theorems 2 and 3: Bisecting Chords and Perpendiculars from the Centre

Theorem 2 states: a straight line drawn from the centre of a circle to bisect a chord (which is not a diameter) is perpendicular to that chord. If M is the midpoint of chord AB and O is the centre, then triangles OAM and OBM share OM, have OA = OB (radii of the same circle) and AM = BM (M is the midpoint, given), so by SSS the triangles are congruent. This makes the two angles at M equal; since they are also supplementary (they form a straight line AMB), each must equal 90 degrees, proving OM is perpendicular to AB.

Theorem 3 is the converse: a perpendicular from the centre of a circle to a chord bisects that chord. If OM is perpendicular to chord AB, triangles OAM and OBM are right triangles sharing hypotenuse... rather, sharing OM, with equal hypotenuses OA = OB (radii), so by the right-triangle hypotenuse-side (H.S.) postulate the triangles are congruent, giving AM = BM. Two useful corollaries follow directly: the perpendicular bisector of any chord of a circle always passes through the centre, and the diameter of a circle passes through the midpoints of any two parallel chords (since both are bisected along the same perpendicular direction from the centre).

9.4 Theorems 4 and 5: Congruent Chords and Distance from the Centre

Theorem 4 states: if two chords of a circle are congruent, then they are equidistant from the centre. Given equal chords AB and CD with perpendiculars OH and OK dropped from the centre O to each, Theorem 3 shows H and K are the midpoints of AB and CD respectively, so AH = CK (half of equal chords are equal). In right triangles OAH and OCK, the hypotenuses OA = OC (radii) and the legs AH = CK, so by the H.S. postulate the triangles are congruent, giving OH = OK -- the two chords are the same perpendicular distance from the centre.

Theorem 5, the converse, states: two chords of a circle which are equidistant from the centre are congruent. The same right triangles OAH and OCK are now congruent because OA = OC (radii) and OH = OK (given equal distances), by the H.S. postulate; this gives AH = CK, and since these are half of chords AB and CD respectively, AB = CD. A classic application of these ideas is proving that the diameter is the longest chord in any circle: since the sum of two sides of a triangle OAB always exceeds the third side, OA + OB > AB, and since OA + OB equals the diameter, the diameter is greater than every other chord.

Important Definitions

What is a circle?

The locus of a moving point in a plane that stays always equidistant from a fixed point called the centre; that constant distance is the radius.

What is the radius of a circle?

The constant distance between the centre of a circle and any point on its circumference.

What is an arc of a circle?

Any portion of the circumference of a circle.

What is a chord of a circle?

A line segment joining any two points on the circumference of a circle.

What is the diameter of a circle?

A chord that passes through the centre of the circle; it bisects the circle into two equal halves.

What is a segment of a circle?

The region of a circle bounded by an arc and its corresponding chord -- the larger region is the major segment and the smaller is the minor segment.

What is a sector of a circle?

The region of a circle bounded by two radii and the arc between them -- the larger region is the major sector and the smaller is the minor sector.

What is the central angle of a circle?

The angle whose vertex is the centre of the circle and whose arms are two radii drawn to the endpoints of an arc.

What does Theorem 2 state about a line bisecting a chord?

A straight line drawn from the centre of a circle to bisect a chord (not a diameter) is perpendicular to that chord.

What does Theorem 4 state about congruent chords?

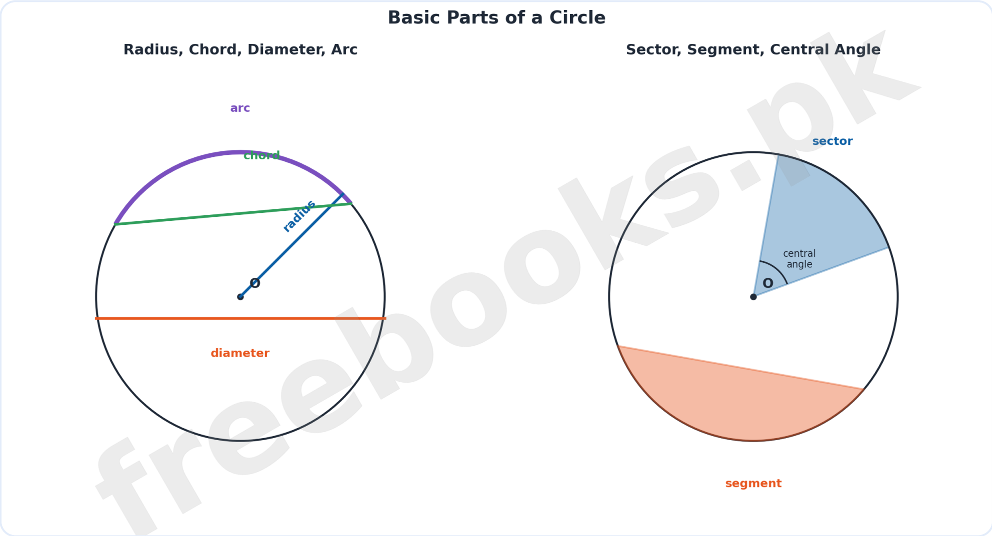
If two chords of a circle are congruent, they are equidistant from the centre of the circle.

Key Facts and Relations

Topic	Key Fact / Relation
Circumference of a circle	C = 2*pi*r
Area of a circle	A = pi*r^2
Theorem 1	One and only one circle can pass through three non-collinear points
Theorem 2	A line from the centre bisecting a chord (not a diameter) is perpendicular to it
Theorem 3 (converse of Theorem 2)	A perpendicular from the centre to a chord bisects the chord
Corollary of Theorem 3	The perpendicular bisector of any chord passes through the centre
Corollary of Theorem 3	The diameter passes through the midpoints of two parallel chords
Theorem 4	Congruent chords of a circle are equidistant from the centre
Theorem 5 (converse of Theorem 4)	Chords of a circle equidistant from the centre are congruent
Longest chord	The diameter is the longest chord of any circle (since OA + OB > AB in triangle OAB)

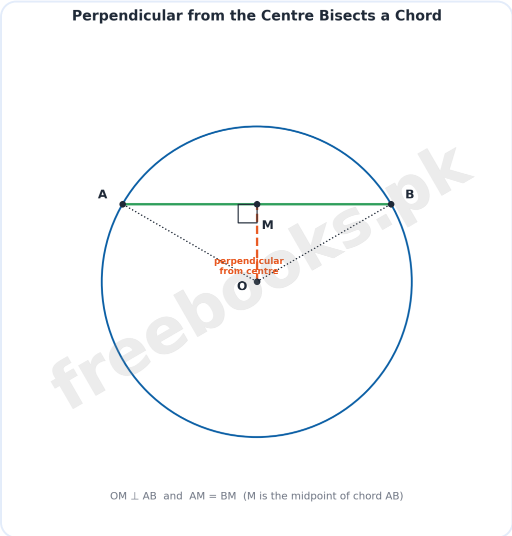
Diagrams

Basic Parts of a Circle: A labelled circle diagram showing the centre, radius, chord, diameter, arc, a major and minor segment, a sector, and the central angle



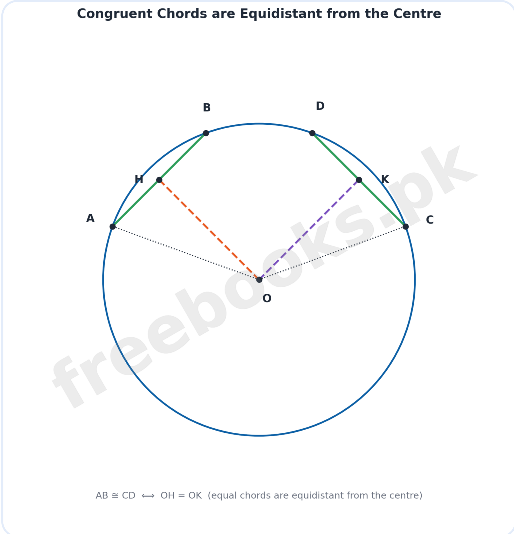
Basic Parts of a Circle

Perpendicular from the Centre Bisects a Chord (Theorems 2 and 3): A circle with centre O and chord AB, showing the perpendicular OM dropped from the centre meeting the chord at its midpoint M



Perpendicular from the Centre Bisects a Chord (Theorems 2 and 3)

Congruent Chords are Equidistant from the Centre (Theorems 4 and 5): A circle with two congruent chords AB and CD, showing the perpendiculars OH and OK dropped from the centre O to each chord, with OH equal to OK



Congruent Chords are Equidistant from the Centre (Theorems 4 and 5)

Short Questions & Answers

Define a chord of a circle.

A chord is a line segment joining any two points on the circumference of a circle.

What is the difference between a sector and a segment of a circle?

A sector is bounded by two radii and an arc, while a segment is bounded by a chord and an arc -- a sector always touches the centre, a segment does not.

State Theorem 2.

A straight line drawn from the centre of a circle to bisect a chord (which is not a diameter) is perpendicular to that chord.

State Theorem 3.

A perpendicular drawn from the centre of a circle to a chord bisects that chord.

State Theorem 4.

If two chords of a circle are congruent, then they are equidistant from the centre of the circle.

Why is the diameter the longest chord of a circle?

Because in triangle OAB formed by the centre and the endpoints of any chord AB, OA + OB > AB (sum of two sides of a triangle exceeds the third), and OA + OB equals the diameter.

Long Questions & Answers

Explain the basic parts of a circle, and state and explain Theorem 1 (one circle through three non-collinear points).

What are the centre, radius, and circumference of a circle?

The centre is the fixed point every point of the circle stays equidistant from; the radius is that constant distance; the circumference is the path traced by the moving point, equal to 2*pi*r.

What is the difference between an arc, a chord, and a diameter?

An arc is a curved portion of the circumference; a chord is a straight line segment joining two points on the circumference; a diameter is a special chord that passes through the centre, bisecting the circle into two equal halves.

What is the difference between a segment and a sector of a circle?

A segment is the region bounded by a chord and its arc (not touching the centre); a sector is the region bounded by two radii and an arc (always touching the centre).

What does Theorem 1 state, and how is it proved?

Theorem 1 states that one and only one circle can pass through three non-collinear points. The perpendicular bisectors of any two of the three connecting segments meet at a single point O that is equidistant from all three points, giving a unique circle centred at O.

How does Theorem 1 explain why every rectangle has a circumscribing circle?

A rectangle's diagonals are equal and bisect each other at a point O, making O equidistant from all four vertices; since any three of those vertices are non-collinear, Theorem 1 guarantees a unique circle through them, and the fourth vertex lies on it too.

Explain Theorems 2-5 relating chords, perpendiculars from the centre, and distance from the centre.

What does Theorem 2 state, and how is it proved?

Theorem 2 states that a line from the centre bisecting a chord (not a diameter) is perpendicular to it. Triangles OAM and OBM are congruent by SSS (OA=OB radii, AM=BM given, OM shared), making the two angles at M equal and supplementary, so each equals 90 degrees.

What does Theorem 3 state, and how does it differ from Theorem 2?

Theorem 3 is the converse of Theorem 2: a perpendicular from the centre to a chord bisects it. It starts from the perpendicularity (given) and proves the bisection (to prove), the opposite direction of logic from Theorem 2.

What are the two corollaries of Theorem 3?

First, the perpendicular bisector of any chord of a circle always passes through the centre. Second, the diameter of a circle passes through the midpoints of any two parallel chords, since both chords are bisected along the same perpendicular direction from the centre.

What does Theorem 4 state about congruent chords?

Theorem 4 states that if two chords of a circle are congruent, they are equidistant from the centre -- proved by showing the right triangles formed by the radii and the half-chords are congruent (H.S. postulate), giving equal perpendicular distances OH = OK.

What does Theorem 5 (the converse of Theorem 4) state, and how is the diameter shown to be the longest chord?

Theorem 5 states that chords equidistant from the centre are congruent. Using the triangle inequality on triangle OAB (OA + OB > AB), and noting OA + OB equals the diameter for any chord AB, it follows that the diameter is longer than every other chord in the circle.

Multiple Choice Questions (MCQs)

The locus of a point in a plane equidistant from a fixed point is called:

A. A radius

B. A circle

C. A circumference

D. A diameter

Answer: B. A circle

Explanation: A circle is defined as the locus of a point that stays equidistant from a fixed centre point.

A chord that passes through the centre of a circle is called:

A. A radius

B. A diameter

C. A sector

D. An arc

Answer: B. A diameter

Explanation: A chord passing through the centre of the circle is specifically called a diameter.

The region bounded by an arc and its chord is called:

A. A sector

B. A segment

C. A radius

D. A circumference

Answer: B. A segment

Explanation: A segment of a circle is the region enclosed between an arc and the chord joining its endpoints.

The region bounded by two radii and the arc between them is called:

A. A sector

B. A segment

C. A diameter

D. A chord

Answer: A. A sector

Explanation: A sector is the region bounded by two radii and the arc intercepted between them.

How many circles can pass through three given non-collinear points?

A. None

B. Exactly one

C. Exactly two

D. Infinitely many

Answer: B. Exactly one

Explanation: By Theorem 1, one and only one circle can pass through three non-collinear points.

A line from the centre of a circle that bisects a chord (not a diameter) is:

A. Parallel to the chord

B. Perpendicular to the chord

C. Equal in length to the chord

D. Twice the chord

Answer: B. Perpendicular to the chord

Explanation: By Theorem 2, a line from the centre bisecting a chord is always perpendicular to that chord.

A perpendicular drawn from the centre of a circle to a chord:

A. Bisects the chord

B. Doubles the chord

C. Is parallel to the chord

D. Passes outside the circle

Answer: A. Bisects the chord

Explanation: By Theorem 3, a perpendicular from the centre to a chord always bisects that chord.

If two chords of a circle are congruent, they are:

A. Parallel

B. Perpendicular to each other

C. Equidistant from the centre

D. Both diameters

Answer: C. Equidistant from the centre

Explanation: By Theorem 4, congruent chords of a circle are always equidistant from the centre.

The longest chord in any circle is:

A. Any random chord

B. The diameter

C. The radius

D. A minor arc

Answer: B. The diameter

Explanation: The diameter is the longest possible chord in a circle, since OA + OB > AB for any other chord AB.

The angle formed at the centre by the two radii to the endpoints of an arc is called:

A. An inscribed angle

B. A central angle

C. A right angle

D. A reflex angle

Answer: B. A central angle

Explanation: The central angle is the angle at the centre of the circle, formed by the two radii drawn to the endpoints of an arc.

Quick Revision Summary

- Circle: locus of a point equidistant from a fixed centre O; radius = constant distance
- Circumference = 2*pi*r | Area = pi*r^2
- Arc: portion of the circumference | Chord: segment joining two points on the circle | Diameter: chord through the centre
- Segment: region bounded by an arc and a chord (major/minor) | Sector: region bounded by two radii and an arc (major/minor)
- Central angle: angle at the centre formed by two radii to the endpoints of an arc
- Theorem 1: one and only one circle passes through three non-collinear points
- Theorem 2: a line from the centre bisecting a chord (not a diameter) is perpendicular to it
- Theorem 3: a perpendicular from the centre to a chord bisects the chord
- Corollaries: perpendicular bisector of a chord passes through the centre; diameter bisects two parallel chords

- Theorem 4: congruent chords are equidistant from the centre | Theorem 5: chords equidistant from the centre are congruent
- The diameter is the longest chord of any circle

Exam Tips

- Always distinguish 'chord' (a straight line segment) from 'arc' (a curved portion of the circumference) -- mixing them up is a very common mistake
- Remember Theorems 2 and 3 are converses of each other -- one starts from 'bisects', the other starts from 'perpendicular'
- Similarly, Theorems 4 and 5 are converses -- one starts from 'congruent chords', the other from 'equidistant from the centre'
- When proving, look for SSS, SAS, or the H.S. (hypotenuse-side) congruence postulate between the two triangles formed by joining the centre to the chord's endpoints
- For 'distance from the centre' problems, that distance is always measured along the PERPENDICULAR from the centre to the chord
- Sketch every problem -- most chord theorems are much easier to see in a diagram than to follow purely from text

Unit 10: Tangent to a Circle

This unit turns from chords to the special line that touches a circle at exactly one point -- the tangent. It begins by distinguishing a tangent from a secant (which cuts the circle at two points), then builds four theorems that describe the tight relationship between a tangent, the radius drawn to its point of contact, and the centre of the circle: a line perpendicular to a radius at its outer end must be tangent there, a tangent is always perpendicular to the radius at the point of contact, the two tangents from any external point are equal in length, and the distance between the centres of two touching circles equals the sum (external contact) or difference (internal contact) of their radii.

These theorems have direct practical uses -- finding unknown tangent lengths, proving lines parallel or perpendicular using tangent properties, and working out the distance between the centres of touching circles -- and they set up the geometry needed for the arcs and inscribed-angle theorems that follow in later units.

Learning Objectives

- Distinguish between a secant and a tangent to a circle, and identify the point of contact
- Prove that a line perpendicular to a radial segment at its outer end point is tangent to the circle at that point
- Prove that the tangent to a circle and the radial segment to the point of contact are perpendicular to each other
- Prove that two tangents drawn to a circle from an external point are equal in length
- Prove that if two circles touch externally or internally, the distance between their centres equals the sum or difference of their radii
- Apply these theorems to solve problems involving tangent lengths, touching circles, and related angles

Key Concepts

10.1 Secant and Tangent Lines

A secant is a straight line that cuts the circumference of a circle in two distinct points. A tangent to a circle, by contrast, is a straight line that touches the circumference at a single point only -- that point is called the point of tangency or the point of contact. Every tangent line lies entirely outside the circle except for this one shared point, while a secant always has a chord of the circle as the segment between its two intersection points.

The length of a tangent to a circle is always measured from the given external point to the point of contact -- this becomes important in Theorem 3, where two tangent lengths from the same external point are compared.

10.2 Theorem 1: Perpendicular at the Outer End of a Radius is Tangent

Theorem 1 states: if a line is drawn perpendicular to a radial segment of a circle at its outer end point, it is tangent to the circle at that point. Given a circle with centre O and radial segment OC, if line AB is perpendicular to OC at C, then for any other point P on AB, triangle OCP has a right angle at C, so the angle at P is acute, making OP greater than OC (the side opposite the larger angle in a triangle is longer). Since OP > OC (the radius), every point on AB except C lies outside the circle -- so AB touches the circle at exactly one point, C, proving it is tangent there.

10.3 Theorem 2: The Tangent is Perpendicular to the Radius

Theorem 2 is essentially the converse: the tangent to a circle and the radial segment joining the point of contact and the centre are perpendicular to each other. If AB is tangent at C and P is any other point on AB, the segment OP (drawn from the centre to P) must cross the circle at some point D, giving OC = OD (radii) and OD < OP (since P is outside), so OC < OP. This makes OC the shortest possible segment from O to the line AB, and the shortest distance from a point to a line is always the perpendicular distance -- so OC is perpendicular to AB.

Corollary: only one perpendicular can be drawn to the radial segment OC at the point C, so one and only one tangent can be drawn to a circle at a given point on its circumference.

10.4 Theorem 3: Two Tangents from an External Point are Equal

Theorem 3 states: two tangents drawn to a circle from a point outside it are equal in length. Given tangents PA and PB from external point P to a circle with centre O, joining O to A, B, and P creates two right triangles OAP and OBP: both have a right angle at the point of contact (Theorem 2), equal hypotenuses OP (shared), and equal legs OA = OB (radii of the same circle) -- so by the right-angle-hypotenuse-side (H.S.) congruence postulate, triangle OAP is congruent to triangle OBP, giving PA = PB.

Corollary: OP is the right bisector of the chord of contact AB. Two classic applications follow: tangents drawn at the two ends of a diameter are always parallel to each other (Example 1), and the tangents drawn at the ends of a chord always make equal angles with that chord (Example 2) -- both proved using the same perpendicularity and congruent-triangle ideas from Theorems 2 and 3.

10.5 Theorems 4(A) and 4(B): Distance Between Centres of Touching Circles

Theorem 4(A): if two circles touch each other externally, the distance between their centres equals the sum of their radii. With circles centred at D and F touching externally at C, and ACB drawn as their common tangent at C, both radial segments CD and CF are perpendicular to the tangent line at C (Theorem 2). Since angle ACD + angle ACF = 90 degrees + 90 degrees = 180 degrees, points D, C, F must be collinear with C between D and F, giving DF = DC + CF.

Theorem 4(B): if two circles touch each other internally, the distance between their centres equals the difference of their radii. The argument is the same, except the common tangent line at C shows that D, C, F are collinear with F between D and C this time, giving DF = DC - FC. A direct application: if three circles touch each other in pairs externally, the perimeter of the triangle formed by joining their centres equals the sum of all three diameters, since each side of the triangle equals the sum of two radii.

Important Definitions

What is a secant of a circle?

A straight line that cuts the circumference of a circle in two distinct points.

What is a tangent to a circle?

A straight line that touches the circumference of a circle at a single point only, called the point of contact or point of tangency.

How is the length of a tangent measured?

From the given external point to the point of contact on the circle.

What does Theorem 1 state about a perpendicular at the outer end of a radius?

If a line is drawn perpendicular to a radial segment of a circle at its outer end point, it is tangent to the circle at that point.

What does Theorem 2 state about a tangent and the radius to its point of contact?

The tangent to a circle and the radial segment joining the point of contact and the centre are perpendicular to each other.

How many tangents can be drawn to a circle at one given point on its circumference?

Exactly one -- only one perpendicular can be drawn to the radial segment at that point.

What does Theorem 3 state about two tangents from an external point?

The two tangents drawn to a circle from a point outside it are equal in length.

What is the corollary of Theorem 3 about the chord of contact?

The segment joining the external point to the centre is the right bisector of the chord of contact joining the two points of tangency.

What does Theorem 4(A) state about externally touching circles?

If two circles touch each other externally, the distance between their centres is equal to the sum of their radii.

What does Theorem 4(B) state about internally touching circles?

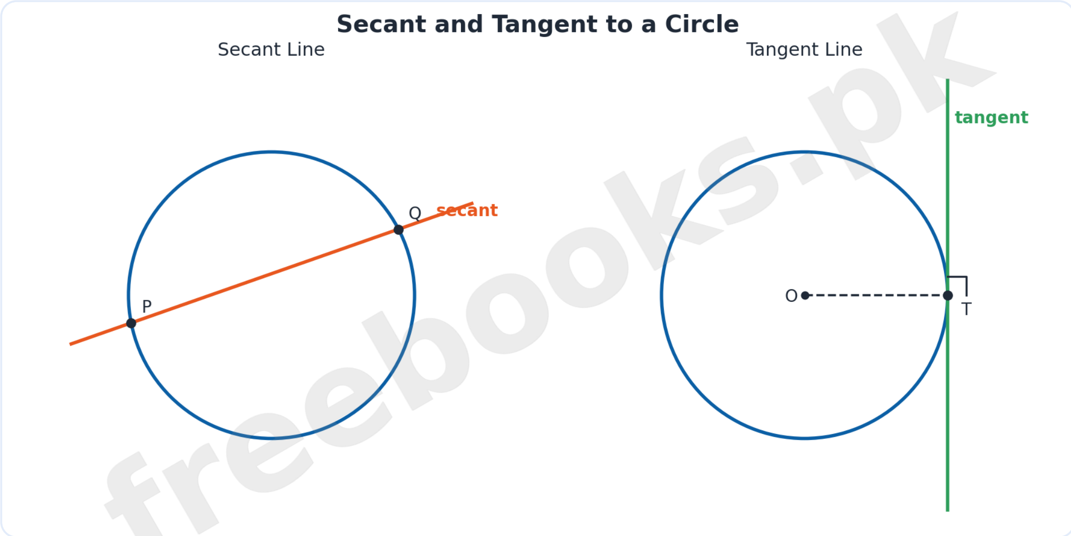
If two circles touch each other internally, the distance between their centres is equal to the difference of their radii.

Key Facts and Relations

Topic	Key Fact / Relation
Secant	A line cutting the circle at two distinct points
Tangent	A line touching the circle at exactly one point (point of contact)
Theorem 1	A line perpendicular to a radius at its outer end point is tangent there
Theorem 2	A tangent and the radius to its point of contact are perpendicular
Corollary of Theorem 2	Only one tangent can be drawn at a given point on a circle
Theorem 3	Two tangents from an external point are equal in length
Corollary of Theorem 3	The line from the external point to the centre right-bisects the chord of contact
Theorem 4(A) -- external contact	Distance between centres = sum of radii (m(DF) = m(DC) + m(CF))
Theorem 4(B) -- internal contact	Distance between centres = difference of radii (m(DF) = m(DC) - m(CF))
Three circles touching in pairs externally	Perimeter of triangle of centres = sum of the three diameters

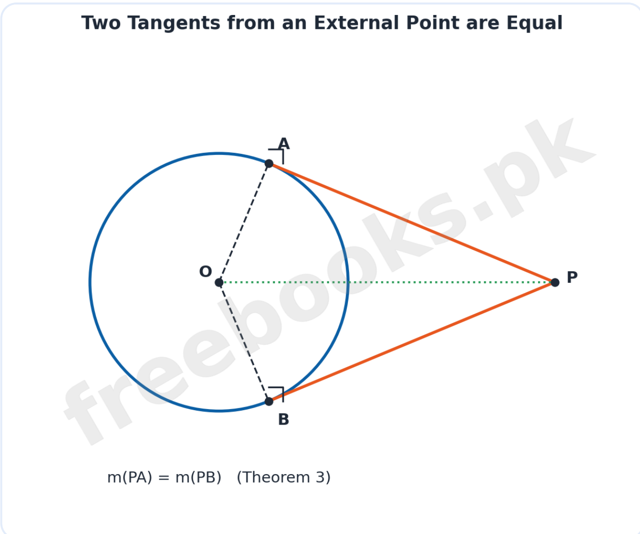
Diagrams

Secant and Tangent to a Circle: A circle showing a secant line cutting the circumference at two distinct points, alongside a tangent line touching the circumference at a single point of contact



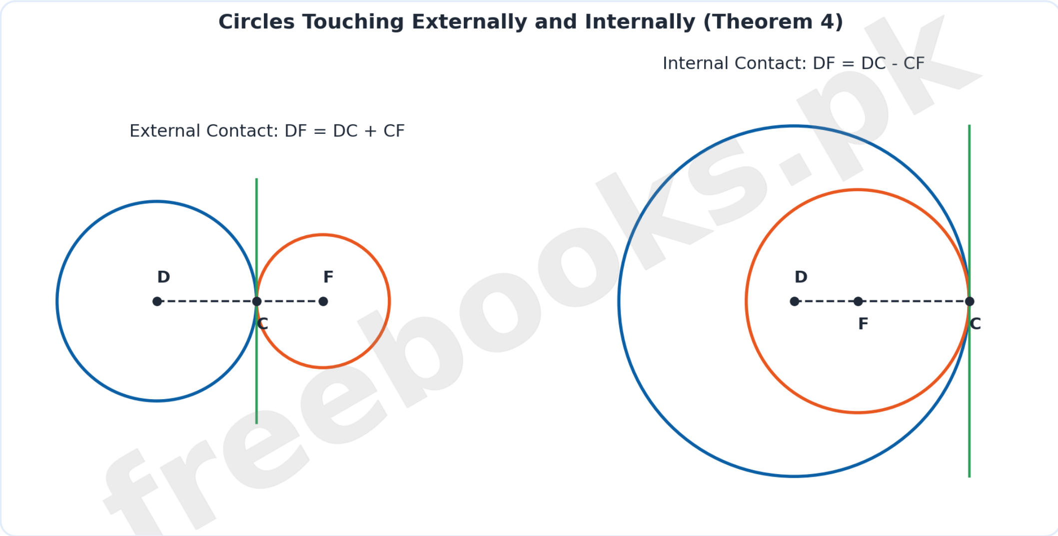
Secant and Tangent to a Circle

Two Tangents from an External Point are Equal (Theorem 3): A circle with centre O and an external point P, showing two tangents PA and PB drawn from P to the circle, with OA and OB as radii perpendicular to the tangents at the points of contact



Two Tangents from an External Point are Equal (Theorem 3)

Circles Touching Externally and Internally (Theorem 4): Two diagrams side by side: circles with centres D and F touching externally at point C with DF = DC + CF, and circles touching internally at point C with DF = DC - CF



Circles Touching Externally and Internally (Theorem 4)

Short Questions & Answers

Define a tangent to a circle.

A tangent is a straight line that touches the circumference of a circle at a single point only, called the point of contact.

How does a secant differ from a tangent?

A secant cuts the circle at two distinct points, while a tangent touches the circle at only one point.

State Theorem 2 of this unit.

The tangent to a circle and the radial segment joining the point of contact and the centre are perpendicular to each other.

State Theorem 3 of this unit.

Two tangents drawn to a circle from a point outside it are equal in length.

What does the corollary of Theorem 3 say about the chord of contact?

The line joining the external point to the centre is the right bisector of the chord of contact joining the two points of tangency.

State Theorem 4(A) about two circles touching externally.

If two circles touch each other externally, the distance between their centres is equal to the sum of their radii.

Long Questions & Answers

Explain the difference between a secant and a tangent, and state and explain Theorems 1 and 2 relating a tangent to the radius at its point of contact.

What is a secant, and what is a tangent?

A secant is a straight line cutting the circumference of a circle at two distinct points. A tangent is a straight line touching the circumference at exactly one point, called the point of contact.

How is the length of a tangent measured?

The length of a tangent to a circle is always measured from the external point from which it is drawn to the point of contact on the circle.

What does Theorem 1 state, and how is it proved?

Theorem 1 states that a line perpendicular to a radial segment at its outer end point is tangent to the circle at that point. Any other point P on the line forms a right triangle with the centre O and the point of tangency C, where the right angle is at C; since the angle at P is acute, OP is longer than OC, placing P outside the circle -- so the line meets the circle only at C.

What does Theorem 2 state, and how is it proved?

Theorem 2 states that a tangent and the radius to its point of contact are perpendicular. For any other point P on the tangent, the segment OP crosses the circle at D, and since $OD < OP$ while $OC = OD$ (radii), OC is the shortest segment from O to the tangent line -- and the shortest distance from a point to a line is always along the perpendicular.

What is the corollary of Theorem 2?

Only one perpendicular can be drawn to a radial segment at its outer end point, so only one tangent line can be drawn to a circle at any given point on its circumference.

State and explain Theorem 3 on tangents from an external point, and Theorems 4(A) and 4(B) on the distance between the centres of touching circles.

What does Theorem 3 state, and how is it proved?

Theorem 3 states that two tangents drawn to a circle from an external point are equal in length. Joining the centre O to the external point P and to both points of contact A and B creates two right triangles OAP and OBP with equal hypotenuses OP, equal legs OA = OB (radii), and right angles at A and B -- congruent by the H.S. postulate, so PA = PB.

What is the corollary of Theorem 3, and what does it help prove about tangents at the ends of a diameter?

The corollary states that the segment from the external point to the centre right-bisects the chord of contact. Using perpendicularity from Theorem 2, tangents drawn at the two ends of a diameter can be shown to both be perpendicular to that diameter, which makes them parallel to each other.

What does Theorem 4(A) state about circles touching externally?

If two circles touch each other externally at point C, the distance between their centres D and F equals the sum of their radii, DF = DC + CF, because the common tangent at C makes angle DCF a straight angle of 180 degrees, placing C between D and F.

What does Theorem 4(B) state about circles touching internally?

If two circles touch each other internally at point C, the distance between their centres equals the difference of their radii, DF = DC - CF, since the smaller circle's centre F lies between D and C along the same line.

How is the sum-of-diameters result used for three mutually externally touching circles?

If three circles touch each other in pairs externally, each side of the triangle formed by joining their centres equals the sum of two radii, so the total perimeter of the triangle equals twice the sum of all three radii -- which is exactly the sum of the three diameters.

Multiple Choice Questions (MCQs)

A line that touches the circumference of a circle at exactly one point is called:

A. A secant
B. A chord
C. A tangent
D. A diameter
Answer: C. A tangent
Explanation: A tangent is defined as a line that touches the circle's circumference at a single point only.

A line that has two points in common with a circle is called:

A. A tangent
B. A radius
C. A secant
D. A diameter
Answer: C. A secant
Explanation: A secant is a line that cuts the circle at two distinct points.

If OT is the radial segment and PTQ is the tangent line at T, then:

A. OT is parallel to PQ
B. OT is perpendicular to PQ
C. OT equals PQ
D. OT bisects the circle
Answer: B. OT is perpendicular to PQ
Explanation: By Theorem 2, the radius to the point of contact is always perpendicular to the tangent line.

Two tangents drawn to a circle from a point outside it are:

A. Half in length
B. Equal in length
C. Double in length
D. Unrelated in length

Answer: B. Equal in length

Explanation: By Theorem 3, the two tangents from the same external point are always equal in length.

A circle can have how many tangents at a single given point on its circumference?

A. Two

B. Three

C. Only one

D. Infinitely many

Answer: C. Only one

Explanation: By the corollary of Theorem 2, only one tangent can be drawn to a circle at a given point.

Tangents drawn at the two ends of a diameter of a circle are:

A. Perpendicular

B. Parallel

C. Collinear

D. Equal to the radius

Answer: B. Parallel

Explanation: Since both tangents are perpendicular to the same diameter, they must be parallel to each other.

A tangent line intersects a circle at:

A. No point

B. A single point

C. Two points

D. Three points

Answer: B. A single point

Explanation: By definition, a tangent touches the circle's circumference at exactly one point.

If two circles touch each other externally, the distance between their centres equals:

A. The difference of their radii

B. The sum of their radii

C. Twice the larger radius

D. Half the sum of their radii

Answer: B. The sum of their radii

Explanation: By Theorem 4(A), external contact means the distance between centres equals the sum of the two radii.

If two circles touch each other internally, the distance between their centres equals:

A. The sum of their radii

B. The difference of their radii

C. Zero

D. The average of their radii

Answer: B. The difference of their radii

Explanation: By Theorem 4(B), internal contact means the distance between centres equals the difference of the two radii.

The length of a tangent to a circle is measured from:

A. The centre to the point of contact

B. The given external point to the point of contact

C. One point of contact to another

D. The circumference to the centre

Answer: B. The given external point to the point of contact

Explanation: The length of a tangent is always measured from the external point it is drawn from to the point where it touches the circle.

Quick Revision Summary

- Secant: line cutting the circle at two points | Tangent: line touching the circle at exactly one point (point of contact)
- Tangent length is measured from the external point to the point of contact
- Theorem 1: a line perpendicular to a radius at its outer end point is tangent to the circle there
- Theorem 2: the tangent and the radius to its point of contact are perpendicular to each other
- Corollary of Theorem 2: only one tangent can be drawn to a circle at a given point
- Theorem 3: two tangents from the same external point are equal in length
- Corollary of Theorem 3: the line from the external point to the centre right-bisects the chord of contact

- Tangents at the two ends of a diameter are parallel; tangents at the ends of a chord make equal angles with it
- Theorem 4(A): circles touching externally -- distance between centres = sum of radii
- Theorem 4(B): circles touching internally -- distance between centres = difference of radii
- Three circles touching in pairs externally: perimeter of the triangle of centres = sum of the three diameters

Exam Tips

- Never confuse 'secant' (two intersection points) with 'tangent' (exactly one) -- draw both if unsure
- Whenever a tangent appears in a proof, immediately mark the right angle between the tangent and the radius at the point of contact -- Theorem 2 is used constantly
- For 'two tangents from a point' problems, look for the H.S. (right-angle-hypotenuse-side) congruence pattern between the two triangles formed
- Remember: external contact adds the radii, internal contact subtracts them -- sketch the touching circles to see which case applies
- The common tangent at the point of contact of two touching circles is always perpendicular to the line joining their centres
- For 'triangle of centres' problems with mutually touching circles, each side is simply the sum of the two relevant radii

Unit 11: Chords and Arcs

This unit connects three quantities associated with a circle -- the chord, the arc it cuts off, and the central angle it subtends -- and proves that any two of these being equal forces the third to be equal too. Four theorems build this equivalence: congruent arcs give equal chords, equal chords give congruent arcs (the converse), equal chords subtend equal central angles, and equal central angles produce equal chords (the converse). All four proofs rely on the same underlying congruent-triangle argument applied to the triangles formed by joining the centre to the endpoints of each chord.

Two useful corollaries follow directly: equal central angles always produce equal sectors, and unequal arcs always subtend unequal central angles. Together these results let a chord length, an arc, or a central angle be substituted for one another freely in later problems -- proving one is often the fastest way to prove all three, and this unit's applied examples (equidistant points, angle bisectors, and perpendicular diameters) show exactly how that substitution is used in practice.

Learning Objectives

- Prove that if two arcs of a circle (or of congruent circles) are congruent, the corresponding chords are equal
- Prove that if two chords of a circle (or of congruent circles) are equal, their corresponding arcs are congruent
- Prove that equal chords of a circle (or of congruent circles) subtend equal angles at the centre
- Prove that if the angles subtended by two chords at the centre are equal, the chords are equal
- State the corollaries relating equal central angles to equal sectors and unequal arcs to unequal central angles
- Apply these theorems to problems involving equidistant points, angle bisectors, and perpendicular diameters

Key Concepts

11.1 Theorem 1: Congruent Arcs Give Equal Chords

Theorem 1 states: if two arcs of a circle (or of congruent circles) are congruent, then the corresponding chords are equal. Given two congruent circles ABCD and A'B'C'D' with centres O and O', and arc ADC congruent to arc A'D'C', joining O to A and C (and O' to A' and C') gives central angles AOC and A'O'C' that are equal, since equal arcs of equal circles subtend equal central angles. Triangles AOC and A'O'C' are then congruent by SAS (OA = O'A' and OC = O'C', being radii of equal circles, with the included angle equal), giving AC = A'C' as a direct consequence. The same argument proves the result within a single circle as well.

11.2 Theorem 2: Equal Chords Give Congruent Arcs (Converse of Theorem 1)

Theorem 2 is the converse: if two chords of a circle (or of congruent circles) are equal, their corresponding arcs are congruent. Starting from AC = A'C' (given) with OA = O'A' and OC = O'C' (radii of equal circles), triangles AOC and A'O'C' are congruent by SSS, giving equal central angles AOC and A'O'C' -- and since arcs correspond directly to their central angles, arc ADC is congruent to arc A'D'C'.

Example: if a point P on the circumference is equidistant from radii OA and OB, then arc AP equals arc BP. Dropping perpendiculars PR and PS from P to the two radii gives two right triangles OPR and OPS sharing hypotenuse OP with equal legs PR = PS, congruent by the H.S. postulate -- so the two central angles at O are equal, making chord AP equal to chord BP, and therefore arc AP congruent to arc BP.

11.3 Theorem 3: Equal Chords Subtend Equal Central Angles

Theorem 3 states: equal chords of a circle (or of congruent circles) subtend equal angles at the centre. Given congruent circles with AC = A'C', the proof uses an indirect argument: assuming angle AOC is not equal to angle A'O'C', construct angle AOC congruent to angle A'O'D' for some other point D' on the second circle. This forces arc AC to equal arc A'D' (Theorem 1's proof direction), and since arc AC also equals arc A'C' (from the equal chords, via Theorem 2), point D' must coincide with C' -- contradicting the assumption and proving angle AOC equals angle A'O'C' after all.

Corollary 1: in congruent circles or in the same circle, if central angles are equal, then the corresponding sectors are equal. Corollary 2: in congruent circles or in the same circle, unequal arcs subtend unequal central angles.

Example: the internal bisector of a central angle in a circle bisects the arc it stands on. If OP bisects angle AOB, triangles OAP and OBP are congruent by SAS (equal radii OA = OB, equal bisected angles, shared side OP), giving AP = BP as chords -- and equal chords correspond to congruent arcs, so arc AP is congruent to arc BP.

11.4 Theorem 4: Equal Central Angles Give Equal Chords (Converse of Theorem 3)

Theorem 4 states: if the angles subtended by two chords of a circle (or congruent circles) at the centre are equal, the chords are equal. Given angle AOC equal to angle A'O'C' in two congruent circles, triangles OAC and O'A'C' are congruent by SAS (OA = O'A' and OC = O'C' as radii of congruent circles, with the included angle equal by hypothesis), giving AC = A'C' directly.

Example: if a pair of diameters of a circle are perpendicular to each other, the lines joining their endpoints in order form a square. With AC and BD as perpendicular diameters, all four central angles at O equal 90 degrees, so by Theorem 4 all four arcs AB, BC, CD, DA are equal, making all four chords (the sides of quadrilateral ABCD) equal in length -- and since each interior angle of the quadrilateral works out to 90 degrees as well, ABCD is a square.

Important Definitions

What does Theorem 1 state about congruent arcs?

If two arcs of a circle (or of congruent circles) are congruent, then the corresponding chords are equal.

What does Theorem 2 state about equal chords?

If two chords of a circle (or of congruent circles) are equal, then their corresponding arcs (minor, major, or semi-circular) are congruent.

What does Theorem 3 state about equal chords and central angles?

Equal chords of a circle (or of congruent circles) subtend equal angles at the centre.

What does Theorem 4 state about equal central angles?

If the angles subtended by two chords of a circle (or congruent circles) at the centre are equal, then the chords are equal.

What is Corollary 1 of Theorem 3?

In congruent circles or in the same circle, if central angles are equal, then the corresponding sectors are equal.

What is Corollary 2 of Theorem 3?

In congruent circles or in the same circle, unequal arcs subtend unequal central angles.

What triangle congruence postulate proves Theorem 1?

SAS -- the two radii and the included central angle are equal, giving equal chords as corresponding sides.

What triangle congruence postulate proves Theorem 2?

SSS -- the two radii and the given equal chords make the triangles congruent, giving equal central angles.

What does it mean for two circles to be congruent?

Two circles are congruent (or equal) when they have equal radii, so one can be superimposed exactly onto the other.

What is a central angle?

The angle whose vertex is the centre of a circle, with its two arms being radii drawn to the endpoints of an arc.

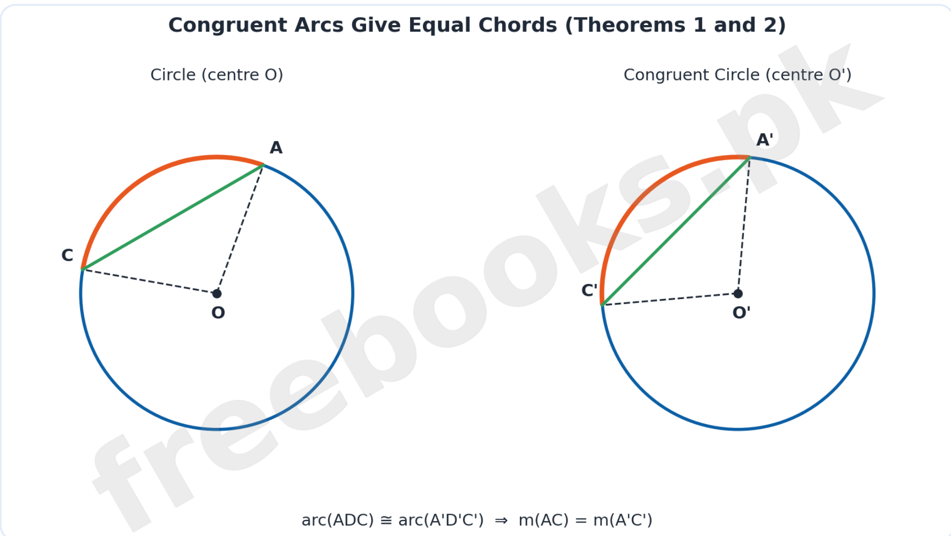
Key Facts and Relations

Topic	Key Fact / Relation
Theorem 1	Congruent arcs give equal corresponding chords
Theorem 2 (converse of Theorem 1)	Equal chords give congruent corresponding arcs

Topic	Key Fact / Relation
Theorem 3	Equal chords subtend equal central angles
Theorem 4 (converse of Theorem 3)	Equal central angles give equal chords
Corollary 1 of Theorem 3	Equal central angles give equal corresponding sectors
Corollary 2 of Theorem 3	Unequal arcs subtend unequal central angles
Congruence used for Theorem 1 and Theorem 4	S.A.S postulate (two radii + included central angle)
Congruence used for Theorem 2 and Theorem 3	S.S.S postulate (two radii + the equal/given chord)
Perpendicular diameters	Two perpendicular diameters divide a circle into four equal arcs, forming a square when endpoints are joined

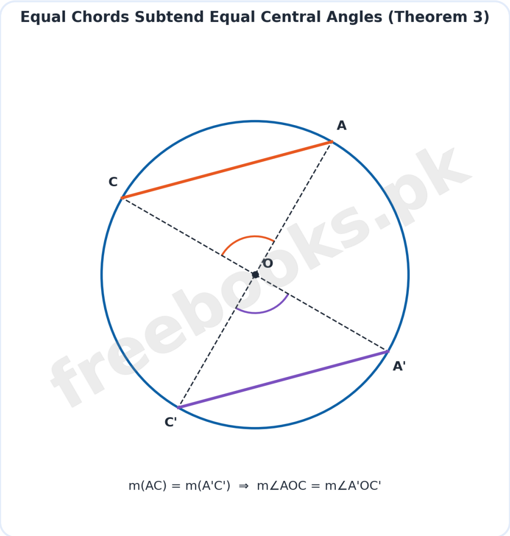
Diagrams

Congruent Arcs Give Equal Chords (Theorems 1 and 2): Two congruent circles with centres O and O', each showing an arc and its corresponding chord, illustrating that congruent arcs ADC and A'D'C' correspond to equal chords AC and A'C'



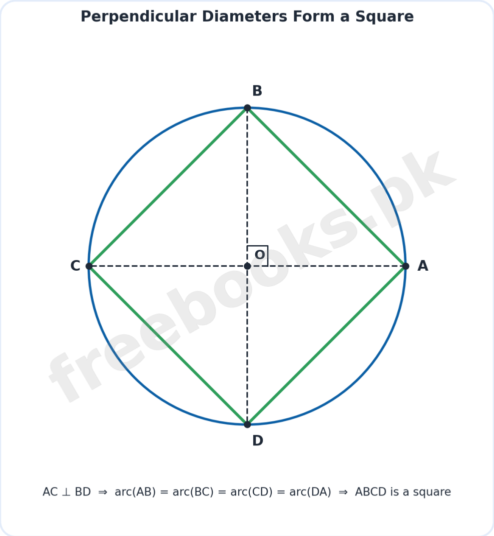
Congruent Arcs Give Equal Chords (Theorems 1 and 2)

Equal Chords Subtend Equal Central Angles (Theorem 3): A circle with centre O showing two equal chords AC and A'C' (drawn within the same circle for comparison) with radii to their endpoints, illustrating that equal chords produce equal central angles AOC and A'OC'



Equal Chords Subtend Equal Central Angles (Theorem 3)

Perpendicular Diameters Form a Square: A circle with centre O and two perpendicular diameters AC and BD, showing the four equal arcs AB, BC, CD, DA and the square ABCD formed by joining their endpoints in order



Perpendicular Diameters Form a Square

Short Questions & Answers

State Theorem 1 of this unit.

If two arcs of a circle (or of congruent circles) are congruent, then the corresponding chords are equal.

State Theorem 2 (the converse of Theorem 1).

If two chords of a circle (or of congruent circles) are equal, then their corresponding arcs are congruent.

State Theorem 3 of this unit.

Equal chords of a circle (or of congruent circles) subtend equal angles at the centre.

State Corollary 1 of Theorem 3.

In congruent circles or in the same circle, if central angles are equal, then the corresponding sectors are equal.

State Corollary 2 of Theorem 3.

In congruent circles or in the same circle, unequal arcs subtend unequal central angles.

Which congruence postulates are used to prove Theorems 1-4 of this unit?

S.A.S is used for Theorems 1 and 4 (radii plus the included central angle); S.S.S is used for Theorems 2 and 3 (radii plus the given equal chord).

Long Questions & Answers

State and explain Theorem 1 and Theorem 2 relating congruent arcs and equal chords, including the point-equidistant-from-radii example.

What does Theorem 1 state, and how is it proved?

Theorem 1 states that congruent arcs of a circle (or of congruent circles) give equal corresponding chords. Joining the centre to the endpoints of each arc gives equal central angles (since equal arcs subtend equal central angles), and with equal radii on both sides, SAS congruence of the two triangles gives equal chords as corresponding sides.

What does Theorem 2 state, and how does it differ in direction from Theorem 1?

Theorem 2 is the converse: equal chords give congruent corresponding arcs. It starts from the chords being equal (given) and uses SSS congruence (equal radii plus the equal chord) to prove the central angles -- and therefore the arcs -- are equal, the reverse direction of logic from Theorem 1.

How is the 'point equidistant from two radii' example proved using Theorem 2?

If point P on the circumference is equidistant from radii OA and OB, the perpendiculars from P to each radius create two right triangles sharing hypotenuse OP with equal legs, congruent by the H.S. postulate. This makes the two central angles at O equal, so chord AP equals chord BP, and by Theorem 2, arc AP is congruent to arc BP.

Why does the same argument prove Theorem 1 within a single circle, not just between two congruent circles?

Since a circle is trivially congruent to itself (equal radii), the same SAS congruence argument applies directly to two arcs and chords within one circle -- the 'congruent circles' version and the 'same circle' version are really the same proof.

State and explain Theorem 3 and Theorem 4 relating equal chords and equal central angles, including the perpendicular-diameters-form-a-square example.

What does Theorem 3 state, and how is it proved?

Theorem 3 states that equal chords subtend equal central angles. The proof is indirect: assuming the central angles are unequal leads to a contradiction (a second point D' would have to coincide with C'), so the central angles must in fact be equal.

What does Theorem 4 state, and how is it proved?

Theorem 4 is the converse: equal central angles give equal chords. Given equal central angles AOC and A'O'C' with equal radii on both sides, SAS congruence of triangles OAC and O'A'C' gives AC = A'C' directly.

What are the two corollaries of Theorem 3?

Corollary 1: equal central angles produce equal corresponding sectors. Corollary 2: unequal arcs subtend unequal central angles -- the natural extension of the equal-angle results to the unequal case.

How does Theorem 4 prove that perpendicular diameters form a square?

Two perpendicular diameters AC and BD create four central angles of 90 degrees each at O. By Theorem 4, equal central angles give equal chords, so all four sides AB, BC, CD, DA of quadrilateral ABCD are equal in length; combined with each interior angle also working out to 90 degrees, ABCD is a square.

What does the angle-bisector example show about bisecting an arc?

If OP bisects central angle AOB, triangles OAP and OBP are congruent by SAS (equal radii, equal bisected angles, shared side OP), giving equal chords AP and BP -- and by Theorem 2, congruent arcs AP and BP, proving the internal bisector of a central angle also bisects the arc it stands on.

Multiple Choice Questions (MCQs)

If two arcs of a circle are congruent, their corresponding chords are:

A. Unequal
B. Equal
C. Perpendicular
D. Parallel
Answer: B. Equal
Explanation: By Theorem 1, congruent arcs always give equal corresponding chords.

If two chords of a circle are equal, their corresponding arcs are:

A. Unequal

B. Perpendicular

C. Congruent

D. Parallel

Answer: C. Congruent

Explanation: By Theorem 2 (the converse of Theorem 1), equal chords produce congruent corresponding arcs.

Equal chords of a circle subtend, at the centre, angles that are:

A. Unequal

B. Supplementary

C. Equal

D. Complementary

Answer: C. Equal

Explanation: By Theorem 3, equal chords always subtend equal angles at the centre of the circle.

If the central angles subtended by two chords are equal, the chords are:

A. Parallel

B. Perpendicular

C. Unequal

D. Equal

Answer: D. Equal

Explanation: By Theorem 4 (the converse of Theorem 3), equal central angles give equal chords.

If central angles in the same circle are equal, the corresponding sectors are:

A. Unequal

B. Equal

C. Perpendicular

D. Overlapping

Answer: B. Equal

Explanation: By Corollary 1 of Theorem 3, equal central angles always give equal corresponding sectors.

Unequal arcs of a circle subtend central angles that are:

A. Equal

B. Complementary

C. Unequal

D. Right angles

Answer: C. Unequal

Explanation: By Corollary 2 of Theorem 3, unequal arcs always subtend unequal central angles.

Theorems 1 and 4 of this unit are proved using which congruence postulate?

A. S.S.S

B. S.A.S

C. A.S.A

D. H.S

Answer: B. S.A.S

Explanation: Theorems 1 and 4 use SAS congruence -- two equal radii with the included central angle equal.

Theorems 2 and 3 of this unit are proved using which congruence postulate?

A. A.S.A

B. H.S

C. S.S.S

D. S.A.S

Answer: C. S.S.S

Explanation: Theorems 2 and 3 use SSS congruence -- two equal radii plus the given equal chord.

Two perpendicular diameters of a circle, when their endpoints are joined in order, always form a:

A. Rectangle

B. Rhombus

C. Square

D. Trapezium

Answer: C. Square

Explanation: Since perpendicular diameters create four equal central angles of

90 degrees, all four chords (sides) are equal and all interior angles are 90 degrees, forming a square.

If a point on a circle's circumference is equidistant from two radii, the arcs cut off from those radii to the point are:

A. Unequal
B. Congruent
C. Perpendicular
D. Parallel

Answer: B. Congruent

Explanation: Equidistance from the two radii makes the two central angles equal, giving equal chords and therefore congruent arcs.

Quick Revision Summary

- Theorem 1: congruent arcs give equal corresponding chords
- Theorem 2 (converse): equal chords give congruent corresponding arcs
- Theorem 3: equal chords subtend equal angles at the centre
- Theorem 4 (converse): equal central angles give equal chords
- Corollary 1: equal central angles give equal sectors
- Corollary 2: unequal arcs subtend unequal central angles
- Theorems 1 and 4 use S.A.S congruence (radii + included central angle)
- Theorems 2 and 3 use S.S.S congruence (radii + the equal chord)
- A point equidistant from two radii cuts off equal chords and congruent arcs from them
- The internal bisector of a central angle also bisects the arc it stands on
- Two perpendicular diameters of a circle, joined at their endpoints, always form a square

Exam Tips

- Chord, arc, and central angle are three interchangeable measures in a circle -- proving any one equal lets you conclude the other two are equal as well
- Watch which direction the proof runs: Theorems 1 and 3 start from arcs/chords and prove angles; Theorems 2 and 4 start from chords/angles and prove arcs -- know which is 'given' and which is 'to prove'

- For SAS-based proofs (Theorems 1 and 4), always check the two radii and the INCLUDED angle between them
- For SSS-based proofs (Theorems 2 and 3), always check the two radii and the given equal chord as the third side
- In 'equidistant from two radii' problems, look for right triangles and the H.S. congruence postulate
- For quadrilateral-in-a-circle problems, converting equal central angles to equal chords (sides) is usually the fastest route to proving a shape regular

Unit 12: Angle in a Segment of a Circle

This unit studies the relationship between a central angle (vertex at the centre) and a circum angle or inscribed angle (vertex on the circumference), both standing on the same arc. The central theorem shows that a central angle is always exactly double the inscribed angle standing on the same arc -- a single fact from which every other result in this unit follows: any two angles in the same segment are equal, the angle in a semicircle is a right angle (with segments larger or smaller than a semicircle giving angles smaller or larger than a right angle), and the opposite angles of a cyclic quadrilateral are always supplementary.

These results are heavily used in circle geometry problems involving inscribed angles, cyclic quadrilaterals, and tangent-chord relationships, and the two worked examples in this unit (equal circles intersecting at a common chord, and a quadrilateral circumscribed about a circle) show how the angle-in-a-segment theorems combine with tangent-length and triangle-congruence ideas from earlier units.

Learning Objectives

- Prove that the central angle of a minor arc is double the inscribed (circum) angle subtended by the corresponding major arc
- Prove that any two angles in the same segment of a circle are equal
- Prove that the angle in a semicircle is a right angle, and relate segment size to whether the inscribed angle is greater or less than a right angle
- Prove that the opposite angles of any cyclic quadrilateral are supplementary
- State the corollaries relating equal arcs to equal inscribed angles
- Apply these theorems to problems involving intersecting circles and quadrilaterals circumscribed about a circle

Key Concepts

12.1 Theorem 1: The Central Angle is Double the Inscribed Angle

Theorem 1 states: the measure of a central angle of a minor arc of a circle is double that of the angle subtended by the corresponding major arc. Given central angle AOC and circum (inscribed) angle ABC standing on arc AC, joining B to the centre O and extending it to D creates two isosceles triangles OAB and OBC. Using the exterior-angle property (an exterior angle equals the sum of the two non-adjacent interior angles) on both triangles and combining the results gives angle AOC = 2 times angle ABC.

Example: a circle has radius root-2 cm, and a chord of length 2 cm divides it into two segments. In triangle OAB, OA squared plus OB squared equals 2 + 2 = 4, which equals AB squared -- so triangle OAB is right-angled at O, meaning the central angle AOB is 90 degrees. By Theorem 1, the circum angle in the larger segment is half the central angle, so angle ACB = 45 degrees.

12.2 Theorem 2: Angles in the Same Segment are Equal

Theorem 2 states: any two angles in the same segment of a circle are equal. If angle ACB and angle ADB are two circum angles standing on the same arc AB, Theorem 1 gives central angle AOB = 2 times angle ACB and also central angle AOB = 2 times angle ADB. Since both expressions equal the same central angle, 2 times angle ACB = 2 times angle ADB, so angle ACB = angle ADB.

12.3 Theorem 3: The Angle in a Semicircle, and Larger/Smaller Segments

Theorem 3 states three related facts: the angle in a semicircle is a right angle; the angle in a segment greater than a semicircle is less than a right angle; and the angle in a segment less than a semicircle is greater than a right angle. All three follow from central angle AOB = 2 times circum angle ACB: when AOB is a straight angle of 180 degrees (a semicircle), ACB must be 90 degrees; when AOB is less than 180 degrees (segment greater than a semicircle), ACB is less than 90 degrees; and when AOB is more than 180 degrees (segment less than a semicircle), ACB is more than 90 degrees.

Corollary 1: the angles subtended by an arc at the circumference of a circle are all equal. Corollary 2: the angles in the same segment of a circle are congruent -- essentially restating Theorem 2 as a direct consequence of Theorem 3's reasoning.

12.4 Opposite Angles of a Cyclic Quadrilateral are Supplementary

This theorem states: the opposite angles of any quadrilateral inscribed in a circle (a cyclic quadrilateral) are supplementary, each pair summing to two right angles (180 degrees). Given cyclic quadrilateral ABCD, angle B is half the central angle standing on arc ADC, and angle D is half the central angle standing on arc ABC; since these two arcs together make up the full circle (a total central angle of 4 right angles, or 360 degrees), angle B + angle D = half of 360 degrees = 180 degrees. The same reasoning shows angle A + angle C = 180 degrees.

Corollary 1: in equal circles or in the same circle, if two minor arcs are equal, the angles inscribed by their corresponding major arcs are also equal. Corollary 2: in equal circles or in the same circle, two equal arcs subtend equal angles at the circumference, and conversely.

Example 1: two equal circles intersect at points A and B; a line through B meets the two circumferences again at P and Q. Since arc ACB equals arc ADB (both stand on the common chord AB in equal circles), the angles they subtend are equal, making triangle APQ isosceles with AP = AQ.

Example 2: quadrilateral ABCD is circumscribed about a circle (each side tangent to it). Dropping perpendiculars from the centre to all four sides and using the equal-tangent-lengths property (two tangents from the same external point are equal) on each vertex shows that AB + CD = BC + DA -- opposite sides of a tangential quadrilateral always have equal sums.

Important Definitions

What is a central angle?

The angle subtended by an arc at the centre of a circle, with its two arms being radii drawn to the endpoints of the arc.

What is a circum angle (inscribed angle)?

The angle subtended by an arc of a circle at a point on its circumference, formed between two chords sharing that common point.

What does Theorem 1 state about the central and inscribed angle?

The measure of the central angle of a minor arc of a circle is double the measure of the inscribed angle subtended by the corresponding major arc.

What does Theorem 2 state about angles in the same segment?

Any two angles in the same segment of a circle are equal.

What is the angle in a semicircle?

A right angle (90 degrees) -- since the central angle on a semicircle is a straight angle of 180 degrees, the inscribed angle is always half of that.

What is a cyclic quadrilateral?

A quadrilateral whose four vertices all lie on the circumference of a single circle.

What does the cyclic quadrilateral theorem state?

The opposite angles of any quadrilateral inscribed in a circle are supplementary -- each pair sums to 180 degrees.

What is a quadrilateral circumscribed about a circle?

A quadrilateral whose four sides are all tangent to a circle inscribed within it.

What is Corollary 1 of Theorem 3?

The angles subtended by the same arc at the circumference of a circle are all equal.

What key property of tangents is used to prove AB + CD = BC + DA for a tangential quadrilateral?

Two tangents drawn to a circle from the same external point are always equal in length.

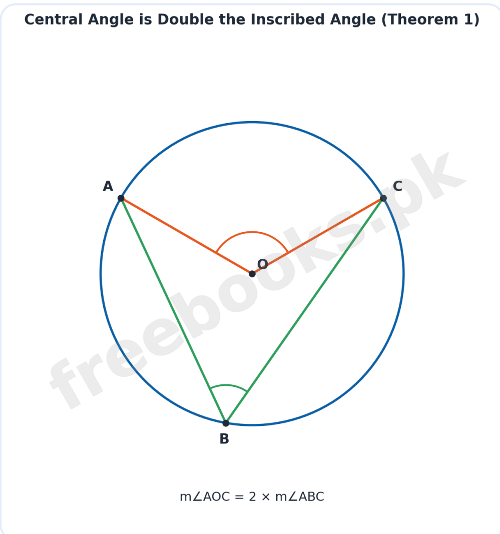
Key Facts and Relations

Topic	Key Fact / Relation
Theorem 1	Central angle = 2 x inscribed (circum) angle on the same arc
Theorem 2	Angles in the same segment of a circle are equal

Topic	Key Fact / Relation
Theorem 3(a)	Angle in a semicircle = 1 right angle (90 degrees)
Theorem 3(b)	Angle in a segment greater than a semicircle < 1 right angle
Theorem 3(c)	Angle in a segment less than a semicircle > 1 right angle
Cyclic quadrilateral theorem	Opposite angles of a cyclic quadrilateral are supplementary (sum = 180 degrees)
Corollary 1 of Theorem 3	Angles subtended by the same arc at the circumference are equal
Tangential quadrilateral (circumscribed about a circle)	AB + CD = BC + DA (sums of opposite sides are equal)

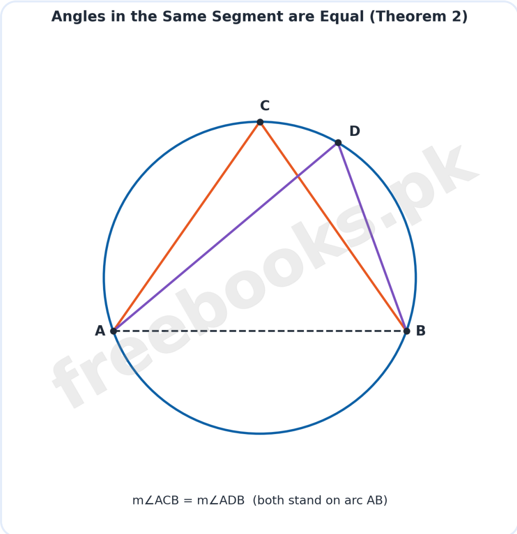
Diagrams

Central Angle is Double the Inscribed Angle (Theorem 1): A circle with centre O showing central angle AOC and inscribed angle ABC standing on the same arc AC, illustrating that the central angle is double the inscribed angle



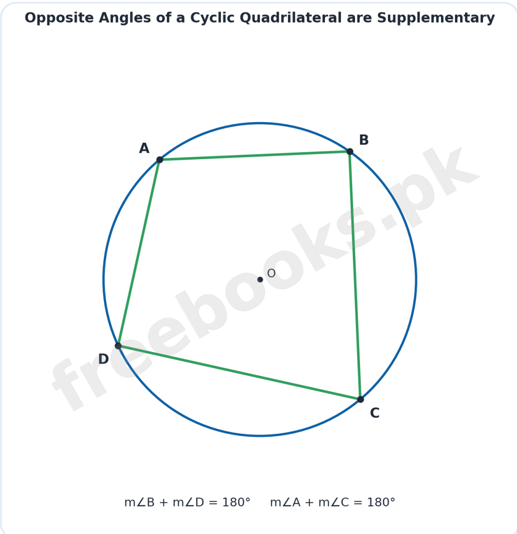
Central Angle is Double the Inscribed Angle (Theorem 1)

Angles in the Same Segment are Equal (Theorem 2): A circle with chord AB and two inscribed angles ACB and ADB standing on the same arc from points C and D in the same segment, showing both angles are equal



Angles in the Same Segment are Equal (Theorem 2)

Opposite Angles of a Cyclic Quadrilateral are Supplementary: A cyclic quadrilateral ABCD inscribed in a circle with centre O, showing that angle B plus angle D equals 180 degrees and angle A plus angle C equals 180 degrees



Opposite Angles of a Cyclic Quadrilateral are Supplementary

Short Questions & Answers

State Theorem 1 of this unit.

The measure of a central angle of a minor arc of a circle is double that of the inscribed (circum) angle subtended by the corresponding major arc.

State Theorem 2 of this unit.

Any two angles in the same segment of a circle are equal.

What is the measure of the angle in a semicircle?

A right angle, that is, 90 degrees.

How does the size of a segment relate to its inscribed angle?

The angle in a segment greater than a semicircle is less than a right angle, while the angle in a segment less than a semicircle is greater than a right angle.

State the theorem about opposite angles of a cyclic quadrilateral.

The opposite angles of any quadrilateral inscribed in a circle are supplementary, each pair summing to 180 degrees.

What does Corollary 1 of Theorem 3 state?

The angles subtended by the same arc at the circumference of a circle are all equal.

Long Questions & Answers

State and explain Theorem 1 (central angle double the inscribed angle) and Theorem 2 (angles in the same segment are equal).

What does Theorem 1 state, and how is it proved?

Theorem 1 states that the central angle of a minor arc is double the inscribed angle on the corresponding major arc. Joining the inscribed angle's vertex to the centre and extending it to the far side of the circle creates two isosceles triangles; using the exterior angle property on both and adding the results shows the central angle equals twice the inscribed angle.

How is Theorem 1 used to find an unknown angle, given a chord length and radius?

If triangle OAB (formed by the centre and the chord's endpoints) satisfies the Pythagorean relationship, its central angle can be shown to be 90 degrees; the inscribed angle in the larger segment is then found as half that central angle, using Theorem 1 directly.

What does Theorem 2 state, and how is it proved?

Theorem 2 states that any two angles in the same segment of a circle are equal. Since both inscribed angles stand on the same arc, Theorem 1 shows each equals half of the same central angle -- so the two inscribed angles must be equal to each other.

Why must both circum angles in Theorem 2 stand on the same arc?

Because Theorem 1 only relates a central angle to an inscribed angle standing on the SAME arc -- if the inscribed angles stood on different arcs, they would correspond to different central angles and there would be no reason for them to be equal.

State and explain Theorem 3 (angle in a semicircle) and the cyclic quadrilateral theorem, including the two worked examples.

What does Theorem 3 state about the angle in a semicircle?

The angle in a semicircle is always a right angle, because the central angle on a semicircle is a straight angle of 180 degrees, and by Theorem 1 the inscribed angle is exactly half of that, giving 90 degrees.

What happens to the inscribed angle in segments larger or smaller than a semicircle?

In a segment greater than a semicircle, the central angle is less than 180 degrees, so the inscribed angle is less than 90 degrees. In a segment less than a semicircle, the central angle exceeds 180 degrees, so the inscribed angle exceeds 90 degrees.

What does the cyclic quadrilateral theorem state, and how is it proved?

The opposite angles of a cyclic quadrilateral are supplementary. Since angle B and angle D are each half of the central angles standing on the two arcs that together make up the whole circle (360 degrees total), their sum is half of 360 degrees, which is 180 degrees.

How does the 'two equal circles intersecting' example use these theorems?

Since the common chord AB subtends equal arcs (and therefore equal inscribed angles) in the two equal circles, triangle APQ formed by the line through B is isosceles, giving AP = AQ.

How does the 'quadrilateral circumscribed about a circle' example use tangent properties?

By dropping perpendiculars from the centre to each of the four tangent sides and using the fact that two tangents from the same external point are equal, the eight tangent segments pair up so that AB + CD equals BC + DA.

Multiple Choice Questions (MCQs)

A central angle standing on the same arc as an inscribed angle is always:

A. Equal to it
B. Half of it
C. Double of it
D. Three times it
Answer: C. Double of it
Explanation: By Theorem 1, a central angle is always double the inscribed angle standing on the same arc.

Any two angles in the same segment of a circle are:

A. Supplementary
B. Complementary
C. Equal
D. Unrelated
Answer: C. Equal
Explanation: By Theorem 2, all inscribed angles in the same segment, standing on the same arc, are equal.

The angle inscribed in a semicircle is always:

A. An acute angle

B. A right angle

C. An obtuse angle

D. A reflex angle

Answer: B. A right angle

Explanation: By Theorem 3, the angle in a semicircle is always exactly 90 degrees, a right angle.

The angle in a segment greater than a semicircle is:

A. Greater than a right angle

B. Equal to a right angle

C. Less than a right angle

D. A straight angle

Answer: C. Less than a right angle

Explanation: By Theorem 3, a segment larger than a semicircle produces an inscribed angle smaller than 90 degrees.

The angle in a segment less than a semicircle is:

A. Less than a right angle

B. Equal to a right angle

C. Greater than a right angle

D. Zero

Answer: C. Greater than a right angle

Explanation: By Theorem 3, a segment smaller than a semicircle produces an inscribed angle greater than 90 degrees.

The opposite angles of a cyclic quadrilateral are:

A. Equal

B. Complementary

C. Supplementary

D. Unrelated

Answer: C. Supplementary

Explanation: The opposite angles of any quadrilateral inscribed in a circle always sum to 180 degrees (supplementary).

A quadrilateral is called cyclic when:

A. All its sides are tangent to a circle

B. All four vertices lie on a circle

C. It has two pairs of parallel sides

D. Its diagonals are equal

Answer: B. All four vertices lie on a circle

Explanation: A cyclic quadrilateral is one whose four vertices all lie on the circumference of a single circle.

For a quadrilateral circumscribed about a circle (tangential quadrilateral), which relation always holds?

A. AB = CD and BC = DA

B. AB + CD = BC + DA

C. AB x CD = BC x DA

D. AB - CD = BC - DA

Answer: B. AB + CD = BC + DA

Explanation: By the equal-tangent-lengths property applied at each vertex, the sums of opposite sides of a tangential quadrilateral are always equal.

If a central angle is 100 degrees, the inscribed angle on the same arc is:

A. 25 degrees

B. 50 degrees

C. 100 degrees

D. 200 degrees

Answer: B. 50 degrees

Explanation: Since the central angle is always double the inscribed angle on the same arc, the inscribed angle is 100/2 = 50 degrees.

Two tangents drawn to a circle from the same external point are always:

A. Perpendicular

B. Parallel

C. Equal in length

D. Unequal in length

Answer: C. Equal in length

Explanation: This equal-tangent-lengths property (from Unit 10) is the key tool used to prove AB + CD = BC + DA for a tangential quadrilateral.

Quick Revision Summary

- Central angle: vertex at the centre | Inscribed (circum) angle: vertex on the circumference
- Theorem 1: central angle = 2 x inscribed angle standing on the same arc
- Theorem 2: any two angles in the same segment of a circle are equal
- Theorem 3: angle in a semicircle = 90 degrees; larger segment gives smaller angle; smaller segment gives larger angle
- Corollary 1: angles subtended by the same arc at the circumference are all equal
- Cyclic quadrilateral: a quadrilateral with all four vertices on a circle
- Opposite angles of a cyclic quadrilateral are supplementary (sum to 180 degrees)
- Tangential quadrilateral (circumscribed about a circle): AB + CD = BC + DA
- In equal circles, equal arcs subtend equal angles at the circumference, and conversely

Exam Tips

- Always identify which angle is central (vertex at O) and which is inscribed (vertex on the circle) before applying Theorem 1
- For 'angles in the same segment' problems, check that both angles stand on the SAME arc -- this is the condition for Theorem 2 to apply
- The angle-in-a-semicircle fact (90 degrees) is one of the most frequently tested results -- watch for a diameter as one side of a triangle inscribed in a circle
- For cyclic quadrilateral problems, remember opposite angles sum to 180 degrees, not adjacent angles
- For tangential quadrilateral problems, look for the equal-tangent-lengths property from Unit 10 applied at all four vertices
- Sketch the central angle and inscribed angle together whenever a problem mixes both -- the factor-of-2 relationship is easy to misapply without a clear diagram

Unit 13: Practical Geometry - Circles

This final unit moves from proving theorems to actually constructing circles and their related figures with a compass and straightedge. It covers locating the centre of a given circle, drawing a circle through three non-collinear points or through part of a given arc, and building the special circles attached to a triangle -- the circumcircle (through all three vertices), the incircle (touching all three sides internally), and the escribed circle (touching one side and the other two sides extended).

The unit then extends to circumscribing and inscribing regular polygons -- equilateral triangles, squares, and regular hexagons -- about and within a given circle, and finally to a large family of tangent constructions: tangents to an arc without knowing its centre, tangents from a point on or outside a circle, tangents meeting at a given angle, and direct (external) and transverse (internal) common tangents to two circles of equal or unequal radii, including circles that already touch or intersect. Every construction in this unit rests on two repeated ideas from earlier units: the perpendicular bisector of a chord passes through the centre, and a tangent is always perpendicular to the radius at its point of contact.

Learning Objectives

- Locate the centre of a given circle and construct a circle through three non-collinear points
- Complete a circle when only part of its circumference is given, both by finding the centre and without finding it
- Circumscribe and inscribe a circle about and in a given triangle, and escribe a circle to a triangle
- Circumscribe and inscribe an equilateral triangle, a square, and a regular hexagon with respect to a given circle
- Construct a tangent to an arc or circle from a point on the circumference, outside it, or on the arc itself, without necessarily knowing the centre

- Construct direct (external) and transverse (internal) common tangents to two circles of equal or unequal radii, including circles that touch or intersect
- Construct a circle touching both arms of an angle, or touching two converging lines and passing through a given point between them

Key Concepts

13.1 Locating a Centre and Constructing Circles Through Points or Arcs

To locate the centre of a given circle, draw any two chords and construct the perpendicular bisector of each; since the perpendicular bisector of any chord always passes through the centre (Unit 9), the two bisectors intersect exactly at the centre O. The same idea, run in reverse, constructs a circle through three given non-collinear points A, B, and C: the perpendicular (right) bisectors of AB and BC meet at a single point O that is equidistant from all three points, and a circle centred at O with that radius passes through all three.

To complete a circle when only part of the circumference (an arc) is given, by finding the centre: mark four points C, D, E, F on the arc, draw chords CD and EF, and construct their perpendicular bisectors -- these meet at the centre O, equidistant from every marked point, and the full circle can then be drawn with radius OA. A second method completes the circle WITHOUT finding the centre, by repeatedly constructing equal external angles between successive equal-length chords along the arc -- each new equal angle and equal chord extends the circle's circumference a little further until it closes up.

13.2 Circles Attached to Triangles and Regular Polygons

The circumcircle of a triangle ABC passes through all three vertices; it is constructed by drawing the perpendicular bisectors of any two sides (say AB and AC) and taking their intersection O as the circumcentre, with circumradius OA = OB = OC. The incircle touches all three sides internally; it is constructed by bisecting two of the triangle's interior angles (say at B and C) to find the incentre O, then dropping a perpendicular OP to any side to get the in-radius OP. The escribed circle (e-circle) touches one side of the triangle externally and the other two sides extended; it is constructed by bisecting two EXTERIOR angles of the triangle to find the e-centre, then dropping a perpendicular to the opposite side for the e-radius.

An equilateral triangle can be circumscribed about a given circle (tangents drawn at three points spaced 120 degrees apart around the circle) or inscribed within it (joining three points on the circle that are each 120 degrees apart, found using arcs of the circle's own radius). A square is circumscribed by drawing two perpendicular diameters and tangents at all four endpoints, or inscribed by simply joining the four endpoints of two perpendicular diameters directly. A regular hexagon is circumscribed or inscribed similarly, using six points spaced 60 degrees apart around the circle -- found by repeatedly stepping off arcs equal to the circle's own radius, since a chord equal to the radius always subtends a 60-degree central angle.

13.3 Tangent Constructions

A tangent can be drawn to an arc without knowing its centre in three cases: when the given point P is the arc's midpoint (construct the perpendicular bisector of the chord through the arc, then erect a right angle at P to that bisector); when P is an endpoint of the arc (use an auxiliary point and an equal-angle construction based on the tangent-chord angle being equal to the inscribed angle in the alternate segment); and when P is outside the arc entirely (using a semicircle construction on segment AP, similar to the construction for a tangent from an external point to a full circle).

When the centre IS known, a tangent from a point P on the circumference is simply the line through P perpendicular to radius OP (Unit 10, Theorem 2). A tangent from a point P outside the circle is constructed by finding the midpoint M of OP, drawing a semicircle on diameter OP centred at M, and joining P to the point T where this semicircle crosses the given circle -- since angle OTP is a right angle (angle in a semicircle), OT is perpendicular to PT, making PT tangent to the circle at T. Two tangents meeting at a specified angle can also be constructed by working backward from the desired angle at the centre.

13.4 Common Tangents to Two Circles, and Circles Touching Lines

Direct (external) common tangents to two EQUAL circles are constructed by drawing parallel diameters perpendicular to the line joining the centres, then joining corresponding endpoints. Transverse (internal) common tangents to two equal circles use a midpoint construction with an auxiliary circle to locate the tangent points on each circle. For two UNEQUAL circles (radii r and r'), direct common tangents use an auxiliary circle of radius r minus r' centred at the larger circle's centre, while transverse common tangents use an auxiliary circle of radius r plus r' -- in both cases combined with a semicircle on the line joining the centres to locate the tangent line's direction, then translated by parallels to the second circle.

For two circles that already touch each other (internally or externally), the single common tangent at the point of contact is simply the line perpendicular to the line joining the two centres at that point (Unit 10, Theorem 2). For two intersecting circles, a tangent construction uses an auxiliary circle of radius r minus r-one together with a midpoint circle, similar in spirit to the unequal-circles case. Finally, a circle can be constructed to touch both arms of a given angle by bisecting the angle (any point on the bisector is equidistant from both arms) and dropping a perpendicular from a chosen point on the bisector to one arm for the radius; the same idea extends to a circle touching two converging lines and passing through a specified point between them. A circle touching three converging (non-parallel, non-concurrent) lines is not generally possible, since three lines in general position do not share a single point equidistant from all three in the required tangential sense.

Important Definitions

How is the centre of a circle located using compass and straightedge?

By drawing two chords and constructing the perpendicular bisector of each -- their point of intersection is the centre, since the perpendicular bisector of any chord always passes through the centre.

What is a circumcircle of a triangle?

The circle passing through all three vertices of a triangle, with its centre (circumcentre) found at the intersection of the perpendicular bisectors of any two sides.

What is an incircle of a triangle?

The circle touching all three sides of a triangle internally, with its centre (incentre) found at the intersection of any two interior angle bisectors.

What is an escribed circle (e-circle) of a triangle?

A circle that touches one side of the triangle externally and the other two sides extended, with its centre (e-centre) found at the intersection of two exterior angle bisectors.

What is a direct (external) common tangent to two circles?

A common tangent line that does not cross the line segment joining the two centres -- it touches both circles on the same side.

What is a transverse (internal) common tangent to two circles?

A common tangent line that crosses the line segment joining the two centres, passing between the two circles.

How is a tangent constructed from a point outside a circle, when the centre is known?

By finding the midpoint of the segment joining the point to the centre, drawing a semicircle on that segment as diameter, and joining the external point to where the semicircle crosses the given circle.

Why does a chord equal in length to the radius subtend a 60 degree central angle?

Because the triangle formed by the two radii and that chord has all three sides equal (each equal to the radius), making it equilateral, and every angle of an equilateral triangle measures 60 degrees.

Why can a circle not generally touch three converging lines?

Because three lines in general position do not share a single point that is simultaneously equidistant from all three in the way required for a common tangent circle to exist.

What must be true of the tangent line at the point where two circles touch each other?

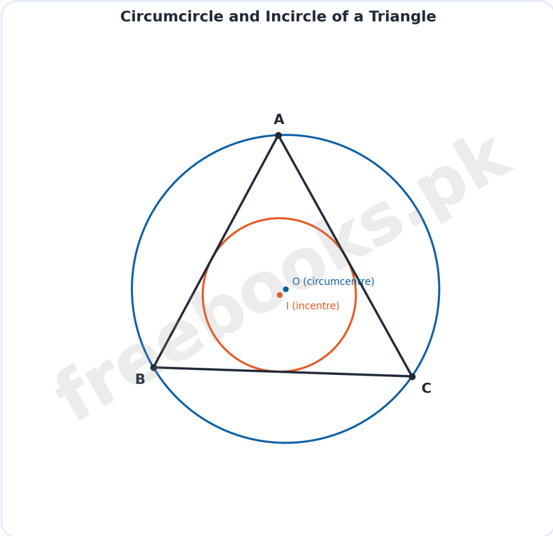
It must be perpendicular to the line joining the two centres at that point of contact.

Key Facts and Relations

Topic	Key Fact / Relation
Circumcircle	Passes through all 3 vertices; centre = intersection of perpendicular bisectors of 2 sides
Incircle	Touches all 3 sides internally; centre = intersection of 2 interior angle bisectors
Escribed circle (e-circle)	Touches 1 side externally, 2 sides extended; centre = intersection of 2 exterior angle bisectors
Inscribed square in a circle	Join the 4 endpoints of 2 perpendicular diameters directly
Inscribed regular hexagon in a circle	6 points spaced 60 degrees apart, found by stepping off chords equal to the radius
Tangent from a point on the circumference	Line through that point, perpendicular to the radius drawn to it
Tangent from a point P outside a circle (centre O)	Semicircle on diameter OP locates the point of tangency T
Direct (external) common tangent, unequal circles ($r > r'$)	Uses an auxiliary circle of radius $r - r'$
Transverse (internal) common tangent, unequal circles	Uses an auxiliary circle of radius $r + r'$
Tangent at the point of contact of two touching circles	Perpendicular to the line joining the two centres at that point

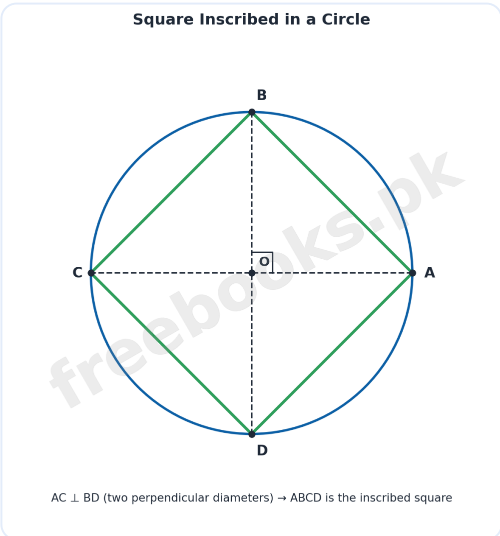
Diagrams

Circumcircle and Incircle of a Triangle: A triangle ABC showing its circumcircle (through all three vertices, centre found from perpendicular bisectors of the sides) and its incircle (touching all three sides, centre found from the angle bisectors)



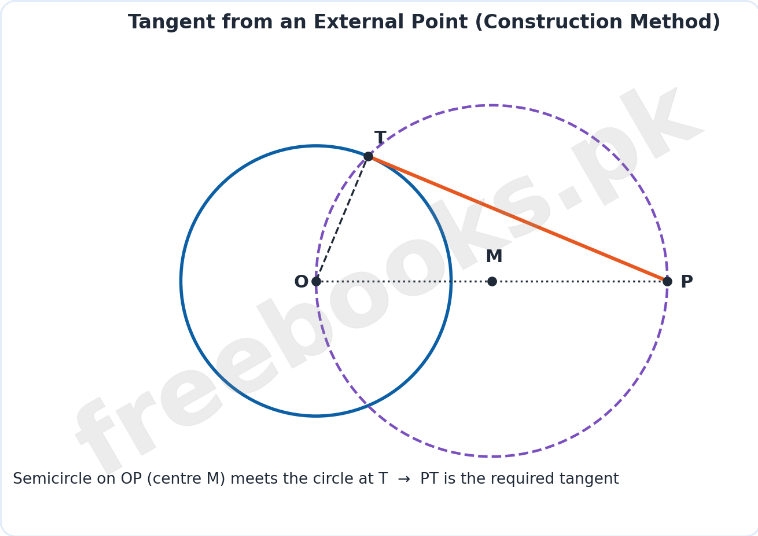
Circumcircle and Incircle of a Triangle

Square Inscribed in a Circle: A circle with centre O and two perpendicular diameters AC and BD, showing the inscribed square ABCD formed by joining the four endpoints in order



Square Inscribed in a Circle

Tangent from an External Point (Construction Method): A circle with centre O and external point P, showing the auxiliary semicircle drawn on diameter OP with centre M, meeting the given circle at T, so that PT is the required tangent



Tangent from an External Point (Construction Method)

Short Questions & Answers

How is the centre of a given circle located using a compass and straightedge?

By drawing two chords and constructing the perpendicular bisector of each; their point of intersection is the centre of the circle.

What is the circumcentre of a triangle, and how is it constructed?

The centre of the circle passing through all three vertices of a triangle, found at the intersection of the perpendicular bisectors of any two sides.

What is the incentre of a triangle, and how is it constructed?

The centre of the circle touching all three sides of a triangle internally, found at the intersection of any two interior angle bisectors.

What is the difference between a direct common tangent and a transverse common tangent?

A direct (external) common tangent does not cross the line joining the two centres, while a transverse (internal) common tangent crosses between the two circles.

How is a square inscribed in a given circle?

By drawing two diameters that bisect each other at right angles, then joining their four endpoints in order to form the square.

Why is it not generally possible to draw a circle touching three converging lines?

Because three lines in general position do not share a single point that can be simultaneously equidistant from all three in the way a common tangent circle requires.

Long Questions & Answers

Describe how to construct the circumcircle, incircle, and escribed circle of a given triangle.

How is the circumcircle of a triangle constructed?

Draw the perpendicular bisectors of any two sides of the triangle; they intersect at the circumcentre O, which is equidistant from all three vertices, giving the circumradius OA = OB = OC.

How is the incircle of a triangle constructed?

Bisect two of the triangle's interior angles; the bisectors meet at the incentre O. Dropping a perpendicular from O to any one side gives the in-radius, and the circle with this radius touches all three sides internally.

How is the escribed circle (e-circle) of a triangle constructed?

Produce two sides of the triangle and bisect the exterior angles formed; these bisectors meet at the e-centre. A perpendicular dropped from the e-centre to the third (un-produced) side gives the e-radius, and the resulting circle touches that side externally and the other two sides extended.

What is the key geometric idea shared by all three constructions?

Each relies on locating a point equidistant from the relevant lines or vertices: the circumcentre is equidistant from the three vertices (via perpendicular bisectors), while the incentre and e-centre are each equidistant from the three sides (via angle bisectors, since every point on an angle bisector is equidistant from the two arms of that angle).

Describe how to construct a tangent from a point outside a circle, and how to construct direct and transverse common tangents to two circles.

How is a tangent constructed from a point P outside a circle with known centre O?

Find the midpoint M of segment OP, then draw a semicircle on OP as diameter with centre M. This semicircle intersects the given circle at a point T; joining P to T gives the required tangent, since angle OTP is 90 degrees (angle in a semicircle), making OT perpendicular to PT.

How are direct (external) common tangents constructed for two equal circles?

Draw diameters of both circles perpendicular to the line joining their centres; joining the corresponding endpoints of these diameters on the same side gives the two direct common tangents.

How are direct common tangents constructed for two UNEQUAL circles?

An auxiliary circle of radius equal to the difference of the two radii is drawn centred at the larger circle's centre; combined with a semicircle on the line joining the centres, this locates points that, when joined through parallels, give the required tangent lines.

How do transverse (internal) common tangents differ in construction from direct tangents?

Transverse tangents cross between the two circles, so their auxiliary circle uses the SUM of the two radii (rather than the difference used for direct tangents), reflecting that the tangent line must pass to the opposite side of each circle.

What construction applies when the two circles already touch each other?

Only one common tangent exists at the point of contact, and it is simply the line perpendicular to the segment joining the two centres, drawn at that point of contact.

Multiple Choice Questions (MCQs)

The centre of a given circle can be located by drawing two chords and constructing:

A. Their midpoints

B. Their perpendicular bisectors

C. Tangents at their endpoints

D. Parallel lines through them

Answer: B. Their perpendicular bisectors

Explanation: The perpendicular bisectors of two chords always intersect at the centre of the circle.

The circumcentre of a triangle is the intersection of:

A. The three medians

B. The three angle bisectors

C. The perpendicular bisectors of two sides

D. The three altitudes

Answer: C. The perpendicular bisectors of two sides

Explanation: The circumcentre is found at the intersection of the perpendicular bisectors of any two sides of the triangle.

The incentre of a triangle is the intersection of:

A. Two interior angle bisectors

B. Two perpendicular bisectors

C. Two medians

D. Two altitudes

Answer: A. Two interior angle bisectors

Explanation: The incentre, equidistant from all three sides, is found at the intersection of two interior angle bisectors.

An escribed circle of a triangle touches:

A. All three sides internally

B. One side internally only

C. One side externally and two sides extended

D. None of the sides

Answer: C. One side externally and two sides extended

Explanation: An escribed (e-)circle touches one side of the triangle externally and the other two sides extended.

A square inscribed in a circle is constructed by joining the endpoints of:

A. One diameter

B. Two perpendicular diameters

C. Three equally spaced radii

D. A single chord

Answer: B. Two perpendicular diameters

Explanation: Two diameters that bisect each other at right angles give four points that, joined in order, form the inscribed square.

A chord equal in length to the radius of its circle subtends a central angle of:

A. 30 degrees

B. 45 degrees

C. 60 degrees

D. 90 degrees

Answer: C. 60 degrees

Explanation: The triangle formed by the two radii and such a chord is equilateral, so the central angle is 60 degrees.

A tangent to a circle from an external point P is constructed using a semicircle drawn on:

A. The radius as diameter

B. The segment OP as diameter

C. A chord as diameter

D. The diameter of the given circle

Answer: B. The segment OP as diameter

Explanation: A semicircle on OP as diameter locates the point of tangency T, since angle OTP must be 90 degrees.

A common tangent that does not cross the line joining the centres of two circles is called:

A. A transverse common tangent

B. An internal common tangent

C. A direct (external) common tangent

D. A secant

Answer: C. A direct (external) common tangent

Explanation: A direct or external common tangent stays on one side and does not cross between the two circles.

For two unequal circles, the auxiliary circle used to construct DIRECT common tangents has a radius equal to:

A. The sum of the two radii

B. The difference of the two radii

C. The average of the two radii

D. Twice the larger radius

Answer: B. The difference of the two radii

Explanation: Direct common tangent construction for unequal circles uses an auxiliary circle of radius equal to the difference of the two radii.

It is generally not possible to construct a circle that touches:

A. Both arms of an angle

B. Two converging lines through a given point

C. Three converging lines

D. A single line at one point

Answer: C. Three converging lines

Explanation: Three lines in general position do not share a single point that can be equidistant from all three, so no such tangent circle generally exists.

Quick Revision Summary

- Locate a circle's centre: draw two chords, construct their perpendicular bisectors, intersection = centre
- Circle through 3 non-collinear points: perpendicular bisectors of 2 connecting segments meet at the centre
- Circumcircle: through all 3 vertices; circumcentre = intersection of perpendicular bisectors of 2 sides
- Incircle: touches all 3 sides; incentre = intersection of 2 interior angle bisectors
- Escribed circle: touches 1 side externally, 2 sides extended; e-centre = intersection of 2 exterior angle bisectors
- Inscribed square: join the 4 endpoints of 2 perpendicular diameters

- Inscribed/circumscribed hexagon: 6 points spaced 60 degrees apart (chord = radius gives 60 degree central angle)
- Tangent from a point on the circle: perpendicular to the radius at that point
- Tangent from an external point P: semicircle on OP as diameter locates the tangent point T
- Direct common tangents: don't cross the centres' line | Transverse common tangents: cross between the circles
- Unequal circles: direct tangents use radius (r - r'); transverse tangents use radius (r + r') for the auxiliary circle
- A circle generally cannot be drawn touching three converging (non-concurrent) lines

Exam Tips

- Every construction in this unit reduces to one of two ideas: perpendicular bisectors locate points equidistant from two POINTS, and angle bisectors locate points equidistant from two LINES
- For circumcircle/incircle/escribed-circle questions, identify first whether you need vertices (perpendicular bisectors) or sides (angle bisectors) to be equidistant from
- The semicircle-on-OP trick for tangent construction works because of the angle-in-a-semicircle theorem from Unit 12 -- keep that connection in mind
- For common-tangent constructions, remember: DIRECT tangents use the DIFFERENCE of radii, TRANSVERSE tangents use the SUM of radii, for the auxiliary circle
- When two circles already touch, there is only one tangent at the point of contact -- perpendicular to the line joining the centres
- Always construct accurately with compass and straightedge and label every intersection point clearly -- construction questions are graded on the steps as much as the final figure